CIRCULAR SUBSETS WITHOUT q-SEPARATION AND POWERS OF LUCAS NUMBERS

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Let n, q, k be integers, $n \geq 1, q \geq 1, k \geq 0$. Consider 1, 2, ..., n displayed in a circle so that n follows 1. Then the integers i, j $(1 \leq j < j \leq n)$ are said to be (circular) *q-separate* if $i+q=j$ or $j+q-n=i$. Let $C_q(n,k)$ denote the number of k-subsets of $\{1, 2, ..., n\}$ without q-separation (no two integers in the subset are q-separate). The (total) number of subsets without q-separation is $C_q(n) = \sum_{k\geq 0} C_q(n,k)$. In this note we prove that

$$C_q(n) = L_m^d, \text{ where } d = \gcd(n,q),\ m = n/d, \tag{1}$$

as follows. Partition the cycle $\{1, 2, ..., n\}$ into d disjoint cycles S_i (reduced modulo n):

$$S_i = \{i, i+q, i+2q, \ldots, i+(m-1)q\},\ 1 \leq i \leq d. \tag{2}$$

The cardinality of each S_i is m, and $C_q(n)$ is equal to the product of the number of subsets of each S_i not containing a pair of consecutive elements. Thus, $C_q(n) = (C_1(m))^a$. But it is an old result that $C_1(n) = L_n$, since $C_1(n)$ can also be interpreted as the number of circular subsets without adjacencies (1 and n are adjacent).

The case $q = 2$ of (1) is

$$C_2(n) = \begin{cases} L_{n/2}^2 & \text{if } n \text{ is even,} \\ L_n & \text{if } n \text{ is odd,} \end{cases} \quad \text{given in [2].}$$

It should be noted that (1) is the special case $x = 1$ of the polynomial identity

$$\sum_{k\geq 0} C_q(n,k)x^k = \left((\alpha(x))^m + (\beta(x))^m\right)^d$$
$$d = \gcd(n,q),\ m = n/d,\ \alpha(x)+\beta(x) = 1,\ ga(x)\beta(x) = -x, \tag{3}$$

established in [2], where the proof involves the same partitioning (2). In the special case $x = 2$, (3) becomes $\sum^{k\geq 0} C_q(n,k)2^k = (2^m + (-1)^m)^d$, $d = \gcd(n,q)$, $m = n/d$.

This has a pleasing combinatorial interpretation, namely, it is the number of 2-colored circular subsets of $\{1, 2, ..., n\}$ without q-separation.

REFERENCES

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