A NOTE ON CONSECUTIVE PRIME NUMBERS

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1. INTRODUCTION

Let p_n denote the n^{th} prime and $d(n) = p_{n+1} - p_n$. Cramer [4], using a probabilistic argument, conjectured that $d(n) = 0((\log(p_n))^2)$. There have been several papers showing that $d(n) = 0(p_n^{\theta})$ (e.g., [7], [8], [9], [10], [12]), for which the value of θ has been reduced to $\frac{11}{20} - \frac{1}{384}$. These papers naturally used sophisticated techniques.

By using the Riemann hypothesis and other properties, one can show $d(n) = 0(p_n^{1/2}(\log(n))^c)$ for some $c > 0$; for example, using the Riemann hypothesis in connection with other assumptions, Heath-Brown & Goldston [6] show that $p_{n+1} - p_n = o(p_n^{1/2}(\log(p_n))^{1/2})$.

As usual, the phrase "almost all n" means that the number of $n \le X$ for which the statement is false is $o(X)$. Now if one is willing to give up the principle of having $d(n) = 0(f(p_n))$ and, instead, demand $d(n) < f(p_n)$ for almost all n, then, as Montgomery showed in [11], for almost all n, the interval $[n, n+n^{1/5+\varepsilon}]$ contains a prime. Harman [5] showed that, for almost all n, the interval $[n, n+n^{1/10+\varepsilon}]$ contains a prime. Once again, sophisticated techniques are used. Better results can be achieved for these types of problems if one can incorporate the moment method found in the papers written by Cheer & Goldston [2], [3].

In this paper we will show that, if $\varepsilon > 0$, $K > 1$, and x is sufficiently large, then the number of indices $n < x$ for which $d(n) \ge K(\log(n))^{1+\varepsilon}$ is less than $x/((K-1)(\log(x))^{\varepsilon})$. Professor Erdös informs me that, if one incorporates Brun's method along with the Prime Number Theorem, then one can establish that the number of indices $n < x$ for which $d(n) > K(\log(n))^{1+\varepsilon}$ is less than $(1-\varepsilon)x/((K(\log(x)^{\varepsilon}))$. The theorems in this paper, though weaker, are elementary and do not depend on Brun's method. We need the following definitions and results. Let $M(x)$ be the set of all positive integers $6 \le n \le x$ for which $d(n) < K(\log(n))^{1+\varepsilon}$ does not hold and let $|M(x)|$ be the cardinality of $M(x)$.

$$\sum_{n \le x} d_n \le p_{x+1} - 2 \tag{1.1}$$

$$p_n \le n(\log(n) + \log\log(n)), \quad n \ge 6. \tag{1.2}$$

It is obvious that (1.1) is a telescoping series. Rosser & Schoenfeld [14] proved (1.2).

2. THEOREMS, LEMMA, AND THEIR PROOFS

Lemma 1: Let $\varepsilon > 0$ and let $M(x)$ be the set of all positive integers $3 \le n \le x$ for which $d(n) < K(\log(n))^{1+\varepsilon}$ does not hold. Let $|M(x)|$ be the cardinality of $M(x)$. Then

$$\sum_{n \in M(x)} (\log(n))^{1+\varepsilon} \ge \int_3^{|M(x)|} (\log(t))^{1+\varepsilon}\,dt > t(\log(t))^{1+\varepsilon} - (1+\varepsilon)t(\log(t))^{\varepsilon}\Big|_3^{|M(x)|}.$$

Theorem 1: Let $\varepsilon > 0$ and

$$K > (x+1)(\log(x+1)+\log\log(x+1))\left[x\left(1-\frac{\log\log^{\varepsilon} x}{\log x}\right)^{\varepsilon}\left(\log(x)\left(1-\frac{\log\log^{\varepsilon} x}{\log x}\right)-1-\varepsilon\right)\right]^{-1}.$$

Let $M(x)$ = the set of positive integers $6 \le n \le x$ such that $d(n)/(\log(n))^{1+\varepsilon} < K$ does not hold. Then $|M(x)| < x/(\log(x))^{\varepsilon}$.

Proof: Let $f(n) = (\log(n))^{1+\varepsilon}$ and let $M'(x) = \{n \ge 6 : n \notin M\}$, that is, the complement of $M(x)$. We have

$$\sum_{6\le n\le x}(f(n)-d(n)) = \sum_{n\in M'(x)}(f(n)-d(n)) + \sum_{n\in M(x)} f(n)(1-d(n)/f(n)). \tag{2.1}$$

If $n \in M(x)$, then $d(n)/f(n) \ge K$, and using this we see that (2.1) becomes

$$\sum_{6\le n\le x}(f(n)-d(n)) \le \sum_{n\in M'(x)}(f(n)-d(n)) + (1-K)\sum_{n\in M(x)} f(n). \tag{2.2}$$

After several manipulations, (2.2) becomes

$$\sum_{n\in M'(x)} d(n) + K\sum_{n\in M(x)} f(n) \le \sum_{6\le n\le x} d(n). \tag{2.3}$$

Dropping the first term on the left-hand side of (2.3) and using (1.1) and (1.2), we now see that (2.3) becomes

$$K\sum_{n\in M(x)} f(n) \le (x+1)(\log(x+1)+\log\log(x+1)). \tag{2.4}$$

Applying Lemma 1 to the left-hand side of (2.4) gives

$$K\int_{6}^{|M(x)|}(\log(t))^{1+\varepsilon}\,dt \le (x+1)(\log(x+1)+\log\log(x+1)). \tag{2.5}$$

From (2.5), we get a contradiction if $|M(x)| \ge x/(\log(x))^{\varepsilon}$. Thus, $|M(x)| < x/(\log(x))^{\varepsilon}$.

Theorem 2: Let $\varepsilon > 0$ and $K > 1$. Let $M(x)$ = the set of positive integers $6 \le n \le x$ such that $d(n)/(\log(n))^{1+\varepsilon} < K$ does not hold. Then, for x sufficiently large, we have

$$|M(x)| < x/((K-1)(\log(x))^{\varepsilon}).$$

Proof: The proof is the same as Theorem 1 up to (2.5). Now

$$K\int_{6}^{|M(x)|}(\log(t))^{1+\varepsilon}\,dt \le (x+1)(\log(x+1)+\log\log(x+1)). \tag{2.6}$$

From (2.6), we get a contradiction if $|M(x)| \ge x/((K-1)(\log(x))^{\varepsilon})$.

3. CONCLUSION

We can now determine that Theorem 2 almost proves Cramer's Conjecture. Let $K > 1$, $\varepsilon = 1$, then for x sufficiently large, by Theorem 2, we have that the number of indices $n < x$ for which $d(n)/(\log(n))^2 < K$ is at least $x - x/((K-1)\log(x))$.

It is also possible to get weaker results without using (1.2). From Ribenboim [13], p. 160, we have $.92129x/\log(x) < \pi(x)$ for $x \geq 30$. If we incorporate this into Theorem 4.7 of Apostol [1], making some minor modifications, we have

$$p_n < 1.62815n(\log(n)) + 0.13347n, \text{ for } p_n \geq 100.$$

Then the following revisions of Theorem 1 and Theorem 2, though not as strong, do not depend on the Prime Number Theorem. The weaker form of Theorem 1 is: suppose $\varepsilon > 0$, $x \geq 100$, and

$$K > 1.62815(x+1)(\log(x+1) + 0.13347)\left[x\left(1 - \frac{\log\log^{\varepsilon} x}{\log x}\right)^{\varepsilon}\left(\log(x)\left(1 - \frac{\log\log^{\varepsilon} x}{\log x}\right) - 1 - \varepsilon\right)\right]^{-1}.$$

Let $M(x)$ = the set of positive integers $6 \leq n \leq x$ such that $d(n)/(\log(n))^{1+\varepsilon} < K$ does not hold. Then $|M(x)| < x/(\log(x))^{\varepsilon}$. The weaker form of Theorem 2 is: suppose $\varepsilon > 0$ and $K > 1$. Let $M(x)$ = the set of positive integers $6 \leq n \leq x$ such that $d(n)/(\log(n))^{1+\varepsilon} < K$ does not hold. Then, for x sufficiently large, we have

$$|M(x)| < 1.6282x/((K-1)(\log(x))^{\varepsilon}).$$

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