REMARK ON A NEW DIRECTION FOR A GENERALIZATION OF THE FIBONACCI SEQUENCE

Krassimir T. Atanassov
Math. Research Lab.—IPACT, P. O. Box 12, Sofia-1113, Bulgaria
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In [5] Peter Hope gives the idea for a Fibonacci-type sequence with the form

$$x_0 = 0,\ x_1 = 1,\ x_{n+2} = a_n x_{n+1} + b_n x_n \quad (n \geq 0),$$

where $\{a_n\}$ and $\{b_n\}$ are given sequences with positive numbers.

Combining this idea with the ideas for a generalization of the Fibonacci sequence from [1], [3], and [6], we shall introduce the new direction for a generalization of the Fibonacci sequence. At the moment, all generalizations of this sequence are "linear." The one proposed here has a "multiplicative" form. The analog of the standard Fibonacci sequence in this form will be

$$x_0 = a,\ x_1 =,\ x_{n+2} = x_{n+1} x_n \quad (n \geq 0),$$

where a and b are real numbers. Directly, it can be seen that, for $n \geq 1$,

$$x_n = a^{f_{n-1}} b^{f_n}.$$

In the case of two (or more) sequences, by analogy with [1], [3], and [6], we shall define the following four schemes.

Scheme I: $$\begin{cases} \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d, \\ \alpha_{n+2} = \beta_{n+1}\beta_n, & (n \geq 0) \\ \beta_{n+2} = \alpha_{n+1}\alpha_n. \end{cases}$$

Scheme II: $$\begin{cases} \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d, \\ \alpha_{n+2} = \alpha_{n+1}\beta_n, & (n \geq 0) \\ \beta_{n+2} = \beta_{n+1}\alpha_n. \end{cases}$$

Scheme III: $$\begin{cases} \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d, \\ \alpha_{n+2} = \beta_{n+1}\alpha_n, & (n \geq 0) \\ \beta_{n+2} = \alpha_{n+1}\beta_n. \end{cases}$$

Scheme IV: $$\begin{cases} \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d, \\ \alpha_{n+2} = \alpha_{n+1}\alpha_n, & (n \geq 0) \\ \beta_{n+2} = \beta_{n+1}\beta_n. \end{cases}$$

Scheme I is analogous to the scheme from [3]; Scheme II is analogous to the scheme from [1] and [6]; Scheme III is analogous to the third scheme from [3]; Scheme IV is a trivial scheme because it contains two Fibonacci sequences in the above-defined "multiplicative form.

Let $\{F_n\}$ be the classical Fibonacci sequence.

The n^{th} terms of these schemes are determined, e.g., as shown in [1], [3], and [6]. We shall give the formulas of the n^{th} terms using the notation from [1] and [3]. These terms are as follows.

Scheme I:

$$\begin{cases} \alpha_{n+2} = a^{\frac{1}{2}\left(F_{n+1}+3\left[\frac{n+2}{3}\right]-n-1\right)} b^{\frac{1}{2}\left(F_{n+1}-3\left[\frac{n+2}{3}\right]+n+1\right)} c^{\frac{1}{2}\left(F_{n+2}-3\left[\frac{n}{3}\right]+n-1\right)} d^{\frac{1}{2}\left(F_{n+2}+3\left[\frac{n}{3}\right]-n+1\right)}, \\ \beta_{n+2} = a^{\frac{1}{2}\left(F_{n+1}-3\left[\frac{n+2}{3}\right]+n+1\right)} b^{\frac{1}{2}\left(F_{n+1}+3\left[\frac{n+2}{3}\right]-n-1\right)} c^{\frac{1}{2}\left(F_{n+2}+3\left[\frac{n}{3}\right]-n+1\right)} d^{\frac{1}{2}\left(F_{n+2}-3\left[\frac{n}{3}\right]+n-1\right)}. \end{cases}$$

Scheme II:

$$\begin{cases} \alpha_{n+2} = a^{\frac{1}{2}(F_{n+1}+\psi(n+2))} b^{\frac{1}{2}(F_{n+1}+\psi(n+5))} c^{\frac{1}{2}(F_{n+2}+\psi(n))} d^{\frac{1}{2}(F_{n+2}+\psi(n+3))}, \\ \beta_{n+2} = a^{\frac{1}{2}(F_{n+1}+\psi(n+5))} b^{\frac{1}{2}(F_{n+1}+\psi(n+2))} c^{\frac{1}{2}(F_{n+2}+\psi(n+3))} d^{\frac{1}{2}(F_{n+2}+\psi(n))}, \end{cases}$$

where ψ is an integer function defined for every $k \geq 0$ by

m	0	1	2	3	4	5
$\psi(6k+m)$	1	0	−1	−1	0	1

(see [1]).

Scheme III:

$$\begin{cases} \alpha_{n+2} = a^{\sigma(n)F_{n+1}} b^{\sigma(n+1)F_{n+1}} c^{\sigma(n+1)F_{n+2}} d^{\sigma(n)F_{n+2}}, \\ \beta_{n+2} = a^{\sigma(n+1)F_{n+1}} b^{\sigma(n)F_{n+1}} c^{\sigma(n)F_{n+2}} d^{\sigma(n+1)F_{n+2}}, \end{cases}$$

where σ is an integer function defined for every $k \geq 0$ by

m	0	1
$\psi(2k+m)$	1	0

Scheme IV:

$$\begin{cases} \alpha_{n+2} = a^{F_{n+1}} c^{F_{n+2}}, \\ \beta_{n+2} = b^{F_{n+1}} d^{F_{n+2}}. \end{cases}$$

The research from [2], [4], [6], [7], and [8] can also be transformed here.

REFERENCES

1. K. Atanassov. "On a Second New Generalization of the Fibonacci Sequence." *The Fibonacci Quarterly* **24.4** (1986):362-65.
2. K. Atanassov. "On a Generalization of the Fibonacci Sequence in the Case of Three Sequences." *The Fibonacci Quarterly* **27.1** (1989):7-10.
3. K. Atanassov, L. Atanassova, & D. Sasselov. "A New perspective to the Generalization of the Fibonacci Sequence." *The Fibonacci Quarterly* **23.1** (1985):21-28.
4. K. Atanassov, J. Hlebarova, & S. Mihov. "Recurrent Formulas of the Generalized Fibonacci and Tribonacci Sequences." *The Fibonacci Quarterly* **30.1** (1992):77-79.
5. P. Hope. "Exponential Growth of Random Fibonacci Sequences." Submitted to *The Fibonacci Quarterly*.
6. J.-Z. Lee & J.-S. Lee. "Some Properties of the Generalization of the Fibonacci Sequence." *The Fibonacci Quarterly* **25.2** (1987):111-17.
7. A. Shannon & R. Melham. "Carlitz Generalizations of Lucas and Lehmer Sequences." *The Fibonacci Quarterly* **31.2** (1993):105-11.
8. W. Spickerman, R. Joyner, & R. Creech. "On the (2, *F*) Generalizations of the Fibonacci Sequence." *The Fibonacci Quarterly* **30.4** (1992):310-14.

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