SOME CONDITIONS FOR "ALL OR NONE" DIVISIBILITY OF A CLASS OF FIBONACCI-LIKE SEQUENCES

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In reference [1], the following theorem has been proved:

Theorem: Let u_n be the general term of a given sequence of integers such that $u_{n+1} = u_{n+1} + u_n$, where u_0 and u_1 are arbitrary integers. Let x be an arbitrary integer other than $-2, -1, 0$ and 1. Let D be any divisor of $x^2 + x - 1$ other than 1. Then the sequence $w_n = xu_{n+1} - u_n$, where $n \geq 0$ is such that:

(a) D divides every w_n;
(b) D divides no w_n.

The aim of this paper is to provide some precise conditions for the "all" situation.

Theorem 1: A necessary, but not sufficient, condition for the sequence with general term $w_n = xu_{n+1} - u_n$ to display the "all" property relative to a given *prime* divisor p of $x^2 + x - 1$ is that the distribution of the residues of (u_n) modulo p be either constant or periodic with period $p-1$.

1) Proof that the condition is necessary:

Let us define the transformation $T_x(u_n)$, for any n, by $T_x(u_n) = xu_{n+1} - u_n$. If $(T_x(u_n))^{(m)}$ denotes the m^{th} iterate of this transformation on (u_n), it is quite easy to prove by induction that, for any n and m:

$$(T_x(u_n))^{(m)} = \sum_{k=0}^{k=m} (-1)^{m+k} \binom{m}{k} (x)^k u_{n+k}.$$

Put $m = p$ in this formula. Since p is prime, the binomial coefficients are all divisible by p, except the two extreme ones ([2], p. 417). Therefore,

$$(T_x(u_n))^{(p)} \equiv x^p u_{n+p} + (-1)^p u_n \pmod{p}.$$

Since no even number can divide $x^2 + x - 1$, p is always an odd prime, and therefore,

$$(T_x(u_n))^{(p)} \equiv x^p u_{n+p} - u_n \pmod{p}$$

for any n. But, since by construction $(T_x(u_n))^{(p)}$ is a linear combination (with integral coefficients) of w_n terms all supposedly divisible by p, this entails

$$x^p u_{n+p} - u_n \equiv 0 \pmod{p}.$$

Since p is prime, $x^p \equiv x \pmod{p}$, and the previous congruence becomes

$$xu_{n+p} - u_n \equiv 0 \pmod{p}.$$

By hypothesis, for any n, $xu_{n+1} - u_n \equiv 0 \pmod{p}$. From the difference of the previous congruences, we obtain

$$x(u_{n+p} - u_{n+1}) \equiv 0 \pmod{p}.$$

Since p and x are relatively prime, this implies that, for any n, $u_{n+p} - u_{n+1} \equiv 0 \pmod{p}$, which proves the necessity of the condition stated above.

Example: In reference [1], we have seen that $w_n = xL_{n+1} - L_n$ displays the property "all" for $x = 2$ and $p = 5$. Therefore, we must have, for any n, $L_{n+5} - L_{n+1} \equiv 0 \pmod{5}$, which property can easily be confirmed.

2) *Proof that the condition is not sufficient:*

To prove this, we shall find an appropriate counter-example deduced from the following lemma.

Lemma: For any x and any prime p dividing $x^2 + x - 1$, the sequence $(w_n) = (xF_{n+1} - F_n)$ displays the "none" property.

Its demonstration is immediate, since $w_0 = x$, and p cannot divide x.

Now, for $x = 7$, we have $x^2 + x - 1 = 55 = 5 * 11$.

But we have $F_{n+11} - F_{n+1} \equiv 0$ (mod 11) for $n = 0$ and $n = 1$. By using the fundamental recurrence property of the Fibonacci numbers, it is then easy to prove this property for any n. However, the above Lemma proves that it is not sufficient to imply the "all" property relative to $p = 11$.

Theorem 2: If, for a sequence $w_n = xu_{n+1} - u_n$, the "all" situation occurs for a nontrivial divisor D of $x^2 + x - 1$, then D divides the quantity $(u_1)^2 - u_0u_2$.

Proof: By definition of D: $x^2 + x - 1 \equiv 0 \pmod{D}$. By multiplying both sides of this congruence by $(u_1)^2$, we obtain $(xu_1)^2 + (xu_1)u_1 - (u_1)^2 \equiv 0 \pmod{D}$. But since $xu_1 \equiv u_0 \pmod{D}$, this is equivalent to $(u_0)^2 + u_0u_1 - (u_1)^2 \equiv 0 \pmod{D}$. And since $(u_1)^2 - u_0u_1 - (u_0)^2 = (u_1)^2 - u_0u_2$, the proof is complete.

This property helps to sharply reduce the number of divisors possible for an "all" situation to occur. For instance, for $u_n = L_n$, $(u_1)^2 - u_0u_2 = -5$. Therefore, 5 is the only possible (positive) divisor of $w_n = xL_{n+1} - L_n$ among those of $x^2 + x - 1$.

But this property of D is not sufficient to warrant the "all" situation, as shown by taking $u_0 = -1$, $u_1 = 4$, *and* $x = 4$. In this case, $x^2 + x - 1 = 19$, so the only possible D is 19 and, on the other hand, $(u_1)^2 - u_0u_2 = 19$. But since $w_0 = 4u_1 - u_0 = 17$, we are in the "none" situation.

REFERENCES

1. Juan Pla. "An 'All or None' Divisibility Property for a Class of Fibonacci-Like Sequences of Integers." *The Fibonacci Quarterly* **32.3** (1994):226-27.
2. Edouard Lucas. *Théorie des Nombres*. Paris, 1891; rpt. Paris: Jacques Gabay, 1991.

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