ON 2-NIVEN NUMBERS AND 3-NIVEN NUMBERS

Tianxin Cai
Department of Mathematics, Hangzhou University, Hangzhou 310028, P. R. China

(Submitted June 1994)

A Niven number [3] is a positive integer that is divisible by the sum of its digits. Various papers have appeared concerning digital sums and properties of the set of Niven numbers. In 1993, Cooper and Kennedy [1] proved that there does not exist a sequence of more than 20 consecutive Niven numbers; they also proved that this bound is the best possible by producing an infinite family of sequences of 20 consecutive Niven numbers. They used a computer to help solve systems of linear congruences, the smallest such sequence they found has **44363342786** digits. In 1994 Grundman [2] generalized the problem to n-Niven numbers with the following definition: For any integer $n \geq 2$, an n-Niven number is a positive integer that is divisible by the sum of its digits in the base n expansion. He proved that no more than $2n$ consecutive n-Niven numbers is possible. He also conjectured that there exists a sequence of consecutive n-Niven numbers of length $2n$ for each $n \geq 2$. In this paper, by solving some congruent equations of higher degree, we obtain the following theorem without the use of a computer.

Theorem: For $n = 2$ or 3, there exists an infinite family of sequences of consecutive n-Niven numbers of length $2n$.

Let $s_n(x)$ denote the digital sum of the positive integer in base n. Consider

$$x = 3^{k_1} + 3^{k_2} + \cdots + 3^{k_8} + 3^3, \quad k_1 > k_2 > \cdots > k_8 > 3,$$

since $s_3(x) = 9$, $s_3(x+1) = 10$, $s_3(x+2) = 11$, $s_3(x-1) = 14$, $s_3(x-2) = 13$, $s_3(x-3) = 12$, the set $\{x-3, x-2, x-1, x, x+1, x+2\}$ is 6 consecutive 3-Niven numbers if and only if the following congruences are satisfied:

$$x_0 + 3 \equiv 0 \pmod{5} \tag{1}$$

$$x_0 + 7 \equiv 0 \pmod{11} \tag{2}$$

$$x_0 + 5 \equiv 0 \pmod{7} \tag{3}$$

$$x_0 + 12 \equiv 0 \pmod{13} \tag{4}$$

$$x_0 \equiv 0 \pmod{4} \tag{5}$$

where $x_0 = 3^{k_1} + 3^{k_2} + \cdots + 3^{k_8}$. Noting that the orders of 3 modulo 5, 11, 7, 13, 4 are 4, 5, 6, 3, 2, respectively, and $[4, 5, 6, 3, 2] = 60$, if the set $\{x-3, x-2, x-1, x, x+1, x+2\}$ is 6 consecutive 3-Niven numbers, then all of the sets $\{x'-3, x'-2, x'-1, x', x'+1, x'+2\}$ with

$$x' = x'(m_1, m_2, \ldots, m_8) = 3^{k_1+60m_1} + 3^{k_2+60m_2} + \cdots + 3^{k_8+60m_8}, m_1, m_2, \ldots, m_8 \geq 0$$

are 6 consecutive 3-Niven numbers.

Note that $3^k \equiv 3 \pmod{4}$ iff $k \equiv 1 \pmod{2}$, $3^k \equiv 1 \pmod{4}$ iff $k \equiv 0 \pmod{2}$. Let x_1 and x_2 denote the number of odd k_i and even k_i, respectively. Then from (5) one has

$$\begin{cases} x_1 + x_2 = 8 \\ 3x_1 + x_2 \equiv 0 \pmod{4} \end{cases} \tag{5'}$$

with particular solutions $(x_1, x_2) = (8, 0), (6, 2), (4, 4)$, or $(2, 6)$.

Similarly, $3^k \equiv 3 \pmod{13}$ iff $k \equiv 1 \pmod 3$, $3^k \equiv 9 \pmod{13}$ iff $k \equiv 2 \pmod 3$, $3^k \equiv 1$ (mod 3) iff $k \equiv 0 \pmod 3$. Let x_1, x_2, and x_3 denote the number of k_i $(1 \le i \le 8)$ in the form $3m+1$, $3m+2$, or $3m$, respectively. Then from (4) one has

$$\begin{aligned} &x_1 + x_2 + x_3 = 8 \\ &3x_1 + 9x_2 + x_3 + 12 \equiv 0 \pmod{13} \end{aligned} \tag{4'}$$

with particular solutions $(1, 7, 0), (3, 0, 5), (4, 3, 1)$, and $(3, 2, 3)$.

Also, $3^k \equiv 3, 2, 6, 4, 5, 1 \pmod 7$ iff $k \equiv 1, 2, 3, 4, 5, 0 \pmod 6$, respectively. Let x_j $(0 \le j \le 5)$ denote the number of k_i $(1 \le i \le 8)$ satisfying $k \equiv j \pmod 6$. Then from (3) one has

$$\begin{aligned} &x_1 + x_2 + x_3 + x_4 + x_5 + x_0 = 8 \\ &3x_1 + 2x_2 + 6x_3 + 4x_4 + 5x_5 + x_0 + 5 \equiv 0 \pmod 7. \end{aligned} \tag{3'}$$

There are many solutions to this system. We find some which also satisfy equations (4′) and (5′). That is,

$$(x_1 + x_3 + x_5, x_2 + x_4 + x_0) = (8, 0), (6, 2), (4, 4), \text{ or } (2, 6);$$
$$(x_1 + x_4, x_2 + x_5, x_3 + x_0) = (1, 7, 0), (3, 0, 5), (4, 3, 1), \text{ or } (3, 2, 3).$$

For example,

$$(x_1, x_2, x_3, x_4, x_5, x_0) = (0, 3, 0, 4, 0, 1), (3, 2, 0, 1, 1, 1), \ldots.$$

Noting that $3^k \equiv 3, 4, 2, 1 \pmod 5$ iff $k \equiv 1, 2, 3, 0 \pmod 4$, respectively, and $3^k \equiv 3, 9, 5, 4, 1$ (mod 11) iff $k \equiv 1, 2, 3, 4, 0 \pmod 5$, respectively. Let x_j $(0 \le j \le 3)$ and x_j $(0 \le j \le 4)$ denote the number of k_i $(1 \le i \le 8)$ satisfying $k \equiv j \pmod 4$ and $k \equiv j \pmod 5$, respectively. Then from equations (1) and (2) one has

$$\begin{aligned} &x_1 + x_2 + x_3 + x_0 = 8 \\ &3x_1 + 4x_2 + 2x_3 + x_4 + 3 \equiv 0 \pmod 5 \end{aligned} \tag{1'}$$

and

$$\begin{aligned} &x_1 + x_2 + x_3 + x_4 + x_0 = 8 \\ &3x_1 + 9x_2 + 5x_3 + 4x_4 + x_0 + 7 \equiv 0 \pmod{11}. \end{aligned} \tag{2'}$$

Let us first consider the solution $(3, 2, 0, 1, 1, 1)$ of equations (3′)-(5′), we make an adjustment so that it also satisfies (1′) and (2′), and obtain

$$x = 1000000001000000011000000000011000000110100,$$

that is,

$$3^3 + 3^5 + 3^6 + 3^{13} + 3^{14} + 3^{25} + 3^{26} + 3^{34} + 3^{43}$$

or

$$328273647965397560259.$$

So the smallest 6 consecutive 3-Niven numbers we obtained has **21** digits. Similarly, from the solution $(0, 3, 0, 4, 0, 1)$ of (3′)-(5′), we obtain $x = 3^3 + 3^4 + 3^{48} + 3^{62} + 3^{64} + 3^{122} + 3^{124} + 3^{182} + 3^{184}$, which has 88 digits.

For the case $n = 2$, we may consider

$$x = 2^{k_1} + 2^{k_2} + 2^{k_3} + 2^4,\ k_1 > k_2 > k_3 > 4.$$

Since $s_2(x)=4$, $s_2(x+1)=5$, $s_2(x-1)=7$, $s_2(x-2)=6$, the set $\{x-2, x-1, x, x+1\}$ is 4 consecutive 2-Niven numbers if and only if

$$\begin{aligned} x_0+1 &\equiv 0 \pmod 5 \\ x_0-1 &\equiv 0 \pmod 7 \\ x_0-2 &\equiv 0 \pmod 3 \end{aligned}$$

are satisfied, where $x_0=2^{k_1}+2^{k_2}+2^{k_3}$. Noting that the orders of 2 modulo 5, 6, 3 are 4, 3, 2, respectively, $[4,3,2]=12$. Therefore, if the set $\{x-2, x-1, x, x+1\}$ is 4 consecutive 2-Niven numbers, all of the sets $\{x'-2, x'-1, x', x'+1\}$ are 4 consecutive 2-Niven numbers, where

$$x'=x'(m_1,m_2,m_3)=2^{k_1+12m_1}+2^{k_2+12m_2}+2^{k_3+12m_3}.$$

We omit the rest of the process. The smallest such sequence we found is $(6222, 6223, 6224, 6225)$ with $6224=2^4+2^6+2^{11}+2^{12}$. Other sequences we found are $(33102, 33103, 33104, 33105)$ with $33104=2^4+2^6+2^8+2^{15}$ and $(53262, 53263, 53264, 53265)$ with $53264=2^4+2^{12}+2^{14}+2^{15}$.

Also we may consider

$$x=2^{k_1}+2^{k_2}+\cdots+2^{k_7}+2^4,\ k_1>k_2>\cdots>k_7>4.$$

The smallest such sequence we found is $(x-2, x-1, x, x+1)$, where

$$x=1100578832=2^4+2^{15}+2^{16}+2^{19}+2^{20}+2^{23}+2^{24}+2^{30}.$$

In principle, this method could be used to find n-Niven numbers of length $2n$ for larger base n. For example, for $n=4$, we may consider $x=4^{k_1}+4^{k_2}+\cdots+4^{k_{15}}+4^{36}$ and, for $n=5$, we may consider $5^{k_1}+5^{k_2}+\cdots+5^{k_{24}}+5^{90}$. But it will be more and more difficult to find a suitable $\{k_1\}$ while n is getting larger.

ACKNOWLEDGMENTS

This project was supported by the NNSFC and NSF of Zhejiang Province.

The author is grateful to the anonymous referee for his or her many useful comments and valuable suggestions.

REFERENCES

1. C. Cooper & R. Kennedy. "On Consecutive Niven Numbers." *The Fibonacci Quarterly* **31.2** (1993):146-51.
2. H. G. Grundman. "Sequences of Consecutive n-Niven Numbers." *The Fibonacci Quarterly* **32.2** (1994):174-75.
3. R. Kennedy, T. Goodman, & C. Best. "Mathematical Discovery and Niven Numbers." *The MATYC Journal* **14** (1980):21-25.

AMS Classification Numbers: 11A07, 11A63

❖❖❖