

ON FIBONACCI HYPERBOLIC TRIGONOMETRY AND MODIFIED NUMERICAL TRIANGLES

Zdzislaw W. Trzaska

Department of Electrical Engineering, Warsaw University of Technology, Warsaw, Poland

Fax: () (48) (2) 6284568; E-mail: trzd@iem.pw.edu.pl

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1. INTRODUCTION

One of the most satisfactory methods for modeling the physical reality consists in arriving at a suitable differential system which describes, in appropriate terms, the features of the phenomenon investigated. The problem is relatively uncomplicated in the finite dimensional setting but becomes very challenging when various partial differential equations, such as the wave, heat, electromagnetic, and other equations, become involved in the more specific description of the system.

When it is difficult or even impossible to obtain an exact solution of the partial differential equations governing a continuous plant, the mathematical model is almost always reduced to a discrete form. Then the plant is represented by an appropriate connection of lumped-parameter elements and it may vibrate only in combinations of a certain set of assumed modes.

In modeling continuous-time systems that are continuous or discrete in space, such classic trigonometric functions as sine, cosine, tangent, and cotangent, as well as corresponding hyperbolic functions, are widely used. As is well known, these functions are based on two irrational numbers: $\pi = 3.14156926\dots$ and $e = 2.7182818\dots$

In this paper we shall be concerned with a new class of hyperbolic functions that are defined on the basis of the irrational number $\phi = \frac{1+\sqrt{5}}{2} \sim 1.618033\dots$, also known as the golden ratio.

We shall introduce new functions called "Fibonacci hyperbolic functions" and show how they result from suitable application of modified numerical triangles. We shall also establish a set of suitable properties of Fibonacci hyperbolic functions such as symmetry, shifting, and links with the classic trigonometric and hyperbolic functions, respectively. Some examples illustrating possible applications of the involved functions in mathematical modeling of physical plants are also presented.

2. THE FIBONACCI TRIGONOMETRY

Recently, studies and applications of discrete functions based on the irrational number $\phi = \frac{1+\sqrt{5}}{2} \sim 1.618033\dots$ have received considerable attention, especially in the theory of recurrence equations, of the Fibonacci sequence, their generalizations and applications (e.g., see [1], [2], [4], [5], [6], [9], and [10]).

In this section we shall present fundamentals of a new class of functions called Fibonacci hyperbolic functions.

Definition 1: Let

$$\psi = 1 + \phi = \frac{3 + \sqrt{5}}{2} \sim 2.618033\dots \quad (1)$$

be given, where ϕ denotes the golden ratio.

For $x \in (-\infty, \infty)$, we define by analogy to the classic hyperbolic functions: chx , shx , thx , $cthx$, continuous functions

$$sFh(x) = \frac{\psi^x - \psi^{-x}}{\sqrt{5}}, \quad cFh(x) = \frac{\psi^{(x+\frac{1}{2})} + \psi^{-(x+\frac{1}{2})}}{\sqrt{5}}, \quad (2)$$

$$tFh(x) = \frac{sFh(x)}{cFh(x)}, \quad ctFh(x) = \frac{cFh(x)}{sFh(x)},$$

as the Fibonacci hyperbolic sine, cosine, tangent, and cotangent, respectively.

Diagrams representing the above-defined Fibonacci sine and cosine are presented in Figure 1. Respective diagrams can easily be established for the Fibonacci tangent and cotangent. They are omitted here for the sake of presentation simplicity.

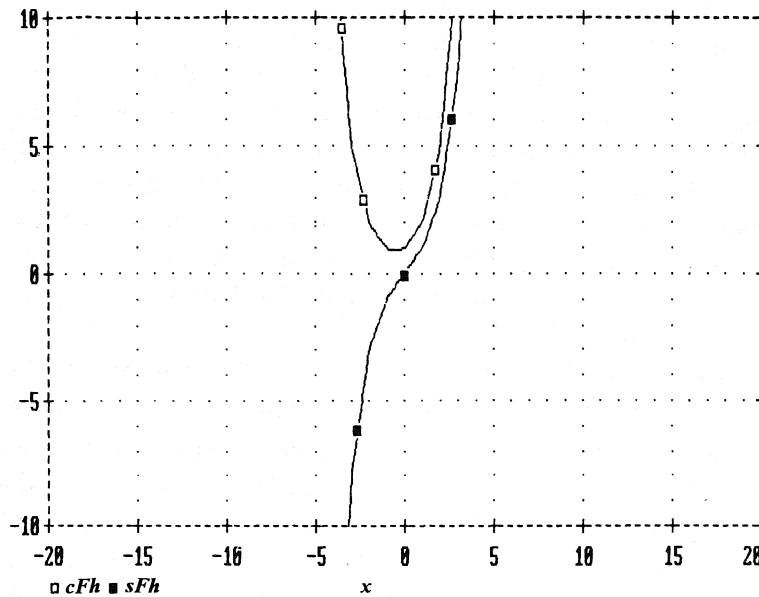


FIGURE 1. Diagrams of $cFh(x)$ and $sFh(x)$

It is worth noting that function $sFh(x)$ is odd-symmetric with respect to the coordinate origin but function $cFh(x)$, while asymmetric with respect to the vertical coordinate $x = 0$, is even-symmetric with respect to $-\frac{1}{2}$.

On the basis of the above definition relations, we are able to establish a set of important properties of Fibonacci hyperbolic functions. In the sequel we shall focus attention on $sFh(x)$ and $cFh(x)$ only.

First, they can be expressed in terms of the golden division ratio ϕ as follows. Using the well-known identity

$$\phi^2 = 1 + \phi \quad (3)$$

and substituting it into expressions (2), we obtain

$$sFh(x) = \frac{\phi^{2x} - \phi^{-2x}}{\sqrt{5}}, \quad cFh(x) = \frac{\phi^{(2x+1)} + \phi^{-(2x+1)}}{\sqrt{5}}. \quad (4)$$

Second, it is easy to demonstrate that when, instead of the continuous independent variable x , we use a discrete variable $k \in I$ (a set of all integer numbers $k = \dots, -2, -1, 0, 1, 2, \dots$) we can express functions $sFh(x)$ and $cFh(x)$ in terms of the corresponding elements of the Fibonacci sequence

$$f(k+1) = f(k) + f(k-1), \quad k = \dots, -3, -2, -1, 0, 1, 2, 3, \dots \quad (5)$$

with $f(0) = 0$ and $f(1) = 1$ as follows:

$$sFh(k) = f(2k), \quad cFh(k) = f(2k+1). \quad (6)$$

Next, applying the well-known Binet formula (see [1], [2]) to the right-hand sides of expressions (6) yields

$$\begin{aligned} sFh(k) &= \frac{1}{2^{2k-1}} \left[\binom{2k}{1} + 5 \binom{2k}{3} + 5 \binom{2k}{5} + \dots + 5^r \binom{2k}{2r+1} + \dots \right] \\ &= \frac{1}{2^{2k-1}} \left[\sum_{r=0}^{\infty} 5^r \binom{2k}{2r+1} \right] \end{aligned} \quad (7)$$

and

$$\begin{aligned} cFh(k) &= \frac{1}{2^{2k}} \left[\binom{2k+1}{1} + 5 \binom{2k+1}{3} + 5 \binom{2k+1}{5} + \dots + 5^r \binom{2k+1}{2r+1} + \dots \right] \\ &= \frac{1}{2^{2k}} \left[\sum_{p=0}^{\infty} 5^p \binom{2k+1}{2p+1} \right]. \end{aligned} \quad (8)$$

Note that the right-hand sides of expressions (6) and (8) do not represent an infinite series but are finite sums, since their general term vanishes for $2k < 2r+1$ and $2k < 2p$, respectively. For instance, at $k = 8$, the first vanishing term corresponds to $2r > 15$ for the $sFh(k)$ and to $p > 8$ for the $cFh(k)$. Thus, the calculations of $sFh(k)$ and $cFh(k)$ ($k \in I$) are reduced to easily computed sums involving simple binomial coefficients, $\binom{n}{m}$.

Finally, it is possible to establish links of the Fibonacci hyperbolic functions $sFh(k)$ and $cFh(k)$ ($k \in I$) with the classic hyperbolic functions $\sinh(x)$ and $\cosh(x)$, but they are based on transition from an expression through its natural logarithm. For this purpose, we calculate the logarithm of the irrational number ϕ , namely,

$$\alpha = \ln \phi = \ln \frac{1+\sqrt{5}}{2} \sim 0.4812118\dots \quad (9)$$

Next, we calculate exponential functions

$$e^{\alpha} = \frac{1+\sqrt{5}}{2} = \phi, \quad e^{-\alpha} = \frac{\sqrt{5}-1}{2} = \phi^{-1}, \quad (10)$$

and the corresponding hyperbolic functions

$$\cosh \alpha = \frac{\sqrt{5}}{2}, \quad \sinh \alpha = \frac{1}{2}. \quad (11)$$

Substituting values (11) raised to the $2k^{\text{th}}$ power into (7) and (8), we get

$$sFh(k) = \frac{2}{\sqrt{5}} \sinh(2k\alpha), \quad cFh(k) = \frac{2}{\sqrt{5}} \cosh[(2k+1)\alpha]. \quad (12)$$

Thus, we have one operation only for calculating $sFh(k)$ or $cFh(k)$, i.e., the multiplication of the known hyperbolic function of the argument $2k\alpha$ or $(2k+1)\alpha$, respectively, by the coefficient $2/\sqrt{5} = 0.8944271\dots$. For example,

$$sFh(8) = \frac{2}{\sqrt{5}} sh(16\alpha) = 0.8944271 \cdot 1103.6922 = 987$$

and

$$cFh(8) = \frac{2}{\sqrt{5}} ch(17\alpha) = 0.8944271 \cdot 1785.5002 = 1597.$$

In a similar manner we can establish links of Fibonacci hyperbolic functions with such trigonometric functions as sine and cosine with respective arguments.

3. PROPERTIES OF FIBONACCI HYPERBOLIC FUNCTIONS

Taking into account the expressions presented in the preceding section, we can derive a set of important properties and relations which come into existence in Fibonacci hyperbolic trigonometry.

First, it is possible to demonstrate on the basis of (6) that the following equalities hold:

$$sFh(-k) = -sFh(k), \quad cFh(-k) = cFh(k-1). \quad (13)$$

Thus, $sFh(k)$ is odd-symmetric with respect to the coordinate origin but $cFh(k)$ is even-symmetric with respect to the vertical line $k = -\frac{1}{2}$. Note that $cFh(-\frac{1}{2}) = \frac{2}{\sqrt{5}} = 0.8944271\dots$, which means that the minimum of $cFh(k)$ appears at $k = -\frac{1}{2}$ and differs from that for the classic hyperbolic $ch(x)$ which equals $\min(\cosh(x))$ at $x = 0$. On the other hand, for $k = 0$, function $cFh(k)$ takes the value $cFh(0) = 1$.

It is easy to prove the remaining important properties of functions $sFh(k)$ and $cFh(k)$. Some of these are given below:

1. $sFh(k) + cFh(k) = sFh(k+1)$,
2. $sFh^2(k) + cFh^2(k) = cFh(2k)$,
3. $cFh^2(k) - sFh^2(k) = 1 + sFh(k)cFh(k)$,
4. $sFh(k) + sFh(\ell) = \sqrt{5}sFh\left(\frac{k+\ell}{2}\right)cFh\left(\frac{k-\ell-1}{2}\right)$,
5. $sFh(k) - sFh(\ell) = \sqrt{5}sFh\left(\frac{k-\ell}{2}\right)cFh\left(\frac{k+\ell-1}{2}\right)$,
6. $cFh(k) + cFh(\ell) = \sqrt{5}cFh\left(\frac{k+\ell}{2}\right)cFh\left(\frac{k+\ell-1}{2}\right)$.

$$7. \operatorname{cFh}(k) - \operatorname{cFh}(\ell) = \sqrt{5} \operatorname{sFh}\left(\frac{k-\ell}{2}\right) \operatorname{sFh}\left(\frac{k+\ell-1}{2}\right),$$

$$8. \operatorname{sFh}(2k) = \sqrt{5} \operatorname{sFh}(k) \operatorname{cFh}\left(k - \frac{1}{2}\right),$$

$$9. \operatorname{cFh}(2k) = \sqrt{5} \operatorname{cFh}(k) \operatorname{cFh}\left(k - \frac{1}{2}\right) + 1,$$

$$10. \operatorname{cFh}(k) \operatorname{cFh}(k-1) - \operatorname{sFh}^2(k) = 1.$$

For the sake of presentation compactness, the corresponding proofs are omitted here. It is worth noting that the above properties also remain valid for continuous arguments $x \in (-\infty, \infty)$ and $y \in (-\infty, \infty)$, respectively.

4. RELATIONSHIPS BETWEEN FIBONACCI HYPERBOLIC FUNCTIONS AND MODIFIED NUMERICAL TRIANGLES

Some advantages in calculating Fibonacci hyperbolic functions follow from the structure and properties of modified numerical triangles (see [5], [9], [10]). To facilitate their demonstration, we shall briefly discuss these triangles and their main characteristics.

The first modified numerical triangle (MNT1) contains elements corresponding to coefficients of polynomials in q defined by the recurrence expression

$$T_{k+1}(q) = (2+q)T_k(q) - T_{k-1}(q), \quad T_0(q) = 1, \quad T_1(q) = 1+q, \quad (14)$$

with q as, in a general case, a complex parameter and $k = 0, \pm 1, \pm 2, \pm 3, \dots$.

Coefficients of the above polynomials for successive values of k belong to MNT1, which takes the form

		MNT1							
	m	0	1	2	3	4	5	6	...
k									
0		1							
1		1	1						
2		1	3	1					
3		1	6	5	1				
4		1	10	15	7	1			
5		1	15	35	28	9	1		
6		1	21	70	84	45	11	1	
...	

The second modified numerical triangle (MNT2) corresponds to polynomials in q defined by the expression

$$P_{k+1}(q) = (2+q)P_k(q) - P_{k-1}(q), \quad P_0(q) = 0, \quad P_1(q) = 1, \quad (15)$$

with $k = 0, \pm 1, \pm 2, \pm 3, \dots$.

Coefficients of these polynomials belong to MNT2, which takes the form

MNT2

	<i>m</i>	0	1	2	3	4	5	6	...
<i>k</i>	0	0							
1	1								
2	2	1							
3	3	4	1						
4	4	10	6	1					
5	5	20	21	8	1				
6	6	35	56	36	10	1			
...			

The above polynomials fulfill a set of important relations, and some examples are as follows:

$$T_k(q) - T_{k-1}(q) = qP_k(q), \quad P_k - P_{k-1} = T_{k-1}(q). \tag{16}$$

It was demonstrated in [9] that, for $q = 1$, the following relations hold:

$$T_k(1) = f(2k + 1), \quad P_k = f(2k), \quad k = 0, \pm 1, \pm 2, \pm 3, \dots \tag{17}$$

Thus, taking into account expressions (6), we have

$$cFh(k) = T_k(1), \quad sFh(k) = P_k(1). \tag{18}$$

It is worth noting that the modified numerical triangles can be used effectively to determine values of corresponding Fibonacci hyperbolic functions.

5. ILLUSTRATION EXAMPLES

Let us now proceed to illustrate possible applications of Fibonacci hyperbolic trigonometry for solving problems arising from biology, physics, or technics. We shall demonstrate these applications through suitable examples.

Example 1: A microwave system usually contains such an essential part as a junction. It consists of two or more microwave components or transmission lines connected together (see [3], [8]). The propagation of electromagnetic signals along each component is described by the transmission line equation,

$$\frac{d^2V}{dx^2} = ZYV, \tag{19}$$

where V is the Laplace transform of the voltage at point $x \in (0, \ell)$ of the space variable in the direction of propagation and Z and Y denote the per unit length impedance and admittance of the line, respectively.

In a general case, the solution for the voltage as a function of time is difficult; for this and other reasons, recourse to an approximate approach is needed. Following this line of reasoning and applying the well-known second-order difference approximation yields

$$V(k + 1) - (2 + q)V(k) + V(k - 1) = 0, \quad V(0) = V_0, \quad V(1) = (1 + q)V(0) \tag{20}$$

with

$$q = Z_0 Y_0, \tag{21}$$

where $Z_0 = Z \cdot h$ and $Y_0 = Y \cdot h$ denote the impedance and admittance per distance $h = \Delta x$ of the space coordinate discretization.

Solving equation (20) with respect to $V(k)$ gives

$$V(k) = T_k(q)V_0 + Z_0 P_k(q)I_0, \tag{22}$$

where V_0 and I_0 denote the Laplace transforms of the voltage and current at $x = 0$, i.e., for $k = 0$.

On the other hand, following the general method of solution of difference equation (20) yields

$$V(k) = q^{-k} [cFh_q(k)V_0 + Z_0 sFh_q(k)I_0], \tag{23}$$

where $sFh_q(k)$ and $cFh_q(k)$ denote generalized Fibonacci hyperbolic sinus and cosinus, respectively. They are defined as follows.

Definition 2: If q denotes, in the general case, a complex parameter, then the following expressions,

$$sFh_q(k) = \frac{1}{\sqrt{q^2 + 4q}} \left[\left(\frac{q + 2 + \sqrt{q^2 + 4q}}{2} \right)^{2k} - \left(\frac{-q - 2 + \sqrt{q^2 + 4q}}{2} \right)^{-2k} \right], \tag{24}$$

$$cFh_q(k) = \frac{1}{\sqrt{q^2 + 4q}} \left[\left(\frac{q + 2 + \sqrt{q^2 + 4q}}{2} \right)^{2k+1} + \left(\frac{-q - 2 + \sqrt{q^2 + 4q}}{2} \right)^{-(2k+1)} \right],$$

define the so-called generalized Fibonacci hyperbolic sinus and cosinus, respectively. Using the above expressions, we can easily establish the generalized Fibonacci hyperbolic tangent and cotangent. For the sake of presentation compactness, corresponding expressions are omitted here.

Thus, comparing solutions (22) and (23) and referring to (24) gives

$$cFh_q(k) = q^k T_k(q), \quad sFh_q(k) = q^k P_k(q). \tag{25}$$

Moreover, it is easily seen that fixing $q = 1$ we obtain the usual Fibonacci hyperbolic functions $cFh(k)$ and $sFh(k)$, so that we have

$$cFh_q(k)|_{q=1} = cFh(k), \quad sFh_q(k)|_{q=1} = sFh(k). \tag{26}$$

Now it is evident that the above presented Fibonacci hyperbolic functions and modified numerical triangles can be very useful for practical problems studies.

Example 2: The filter design problem at microwave frequencies, where distributed parameter elements must be used, is extremely complicated, and no complete theory or synthesis procedure exists for solving the problem. The complex behavior of microwave circuit elements makes it impossible to develop a general and complete synthesis procedure [7]. However, a procedure based on the Fibonacci hyperbolic trigonometry appears as useful technique for studies of microwave filters. The effect of lossy elements or quarter-wave transformers can easily be considered.

The latter case is represented by the network shown in Figure 2. It contains a number of quarter-wave transformers loaded by the lumped parameter elements characterized by impedance Z_1 . The voltage and current distributions along the system are described by the matrix equation

$$\begin{bmatrix} U(k+1) \\ I(k+1) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} U(k) \\ I(k) \end{bmatrix}, \quad k = 0, 1, 2, \dots, \quad (27)$$

where a , b , c , and d denote, in a general case, complex parameters fulfilling the relation

$$ad - bc = 1. \quad (28)$$

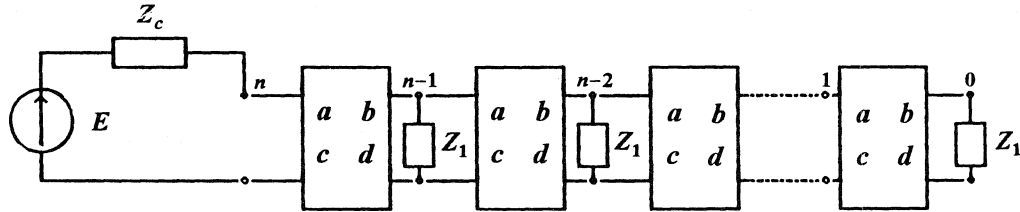


FIGURE 2. Ladder of Two Ports

In the sequel we shall limit our attention to a system having the following parameters:

$$a = 1, \quad b = -jZ_c, \quad c = \frac{-j}{Z_c}, \quad d = 0, \quad (29)$$

where Z_c is the characteristic impedance of each one of two port elements in the system and $j = \sqrt{-1}$.

Introducing characteristic parameter

$$p = \frac{Z_1}{Z_c}, \quad (30)$$

and solving equation (27) with respect to $U(k)$, $k = 0, 1, 2, \dots$, we get the second-order difference equation with complex coefficients, that is,

$$U(k+1) + jpU(k) + U(k-1) = 0, \quad U(0) = U_0, \quad U(1) = -jpU(0). \quad (31)$$

Now, comparing respective coefficients in equations (22) and (31) yields

$$U(2k) = (-j)^{2k} T_k(p^2), \quad U(2k+1) = (-j)^{2k+1} P_k(p^2), \quad (32)$$

where $T_k(x)$ and $P_k(x)$ are the polynomials in $x = p^2$ with coefficients from MNT1 and MNT2, respectively.

Taking into account the relationship between Fibonacci hyperbolic functions and polynomials $T_k(x)$ and $P_k(x)$, we can transform relations (32) into the following forms:

$$U(2k) = \left(\frac{1}{jp^2} \right)^{2k} cFh_{p^2}(k), \quad U(2k+1) = \left(\frac{1}{jp^2} \right)^{2k+1} sFh_{p^2}(k), \quad \text{with } k = 0, 1, 2, \dots \quad (33)$$

Thus, a set of suitable expressions has been established which gives much more facility and improvement with respect to up-to-date available ones in the design of microwave filters. It must be stressed that no assumption has been made on the lumped parameter elements; therefore, the presented approach is quite general.

Example 3: One of the fundamental problems in botany lies in suitable descriptions of leaf growings [12]. The geometry of leaf growing is characterized by a spiral-symmetry structure. Bio-organisms draw images on the surface of the leaves forming left- and right-turning spiral lines with crossings at respective points. The symmetry order of the leaf-grilles are determined by a number of spiral lines in respective patterns. During leaf growing, these spiral lines can be transformed into moving hyperboles with cross-points determined by the coordinates expressed in terms of Fibonacci hyperbolic functions as follows:

$$u_k = a \cdot sFh(k), \quad u_{k-1} = a \cdot cFh(k-1), \quad (34)$$

where $k = 0, 1, 2, \dots$ is the index of the cross-point in the leaf-grille and a denotes the scale coefficient of the moving hyperbole with respect to parameters of a unit hyperbole.

If the grille is square, then the coordinates of the cross-points take integer values that fulfill the relation

$$u_{k+1}u_k - u_{k+1}^2 + u_k^2 = \text{const}. \quad (35)$$

The structure-symmetry order of the logarithmic grille is determined by the parameter

$$q_\ell = q^D, \quad (36)$$

where q denotes the similarity coefficient and D is the angle divergence.

For tree foliage, leaf growing is determined in terms of the Fibonacci sequence and fulfills the equation

$$|f^2(k) + f(k)f(k+1) - f^2(k+1)| = 1, \quad (37)$$

and at the limit $k \rightarrow \infty$, the angle divergence is equal to

$$\lim_{k \rightarrow \infty} D = \frac{\sqrt{5}-1}{2} = \phi^{-1} \sim 0.618033\dots \quad (38)$$

Other cases of leaf growing are governed by similar expressions. Following a more general line of reasoning, it is possible to prove that there are general principles in pattern formation on the plants.

6. CONCLUSIONS

In this paper we have presented the new ideas and concepts concerning hyperbolic trigonometry. It has been shown that many problems appearing in mathematical modeling of physical plants can be solved successfully by applying such new functions as Fibonacci hyperbolic sine and/or cosine. The concepts presented in this paper have the following features: a) they produce analytic expressions for both continuous and discrete arguments; b) in the discrete case, there exist respective links with the classic Fibonacci sequence; c) important simplifications in calculus can be achieved by using modified numerical triangles.

The application of Fibonacci hyperbolic functions has been illustrated by suitable examples.

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