A FAMILY OF 4-BY-4 FIBONACCI MATRICES

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1. AIM OF THE PAPER

Fibonacci matrices are matrices the entries of the powers of which are related to the Fibonacci numbers F_n and/or the Lucas numbers L_n. The most celebrated among them are the 2-by-2 matrix **Q** (e.g., see [6, p. 65]) first studied by C. H. King in [7, pp. 11-27], and the 3-by-3 matrix **R** (e.g., see [1, p. 26]).

Consider the m-by-m tridiagonal symmetric Toeplitz matrix $\mathbf{S}_m(x, y)$ defined as

$$\mathbf{S}_m(x,y) = \begin{bmatrix} x & y & 0 & 0 & \cdots & 0 & 0 & 0 \\ y & x & y & 0 & \cdots & 0 & 0 & 0 \\ 0 & y & x & y & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & y & x & y \\ 0 & 0 & 0 & 0 & \cdots & 0 & y & x \end{bmatrix}, \tag{1.1}$$

where x and y are arbitrary quantities.

In this paper we show how, for certain (integral) values of x and y, the matrices $\mathbf{S}_4(x, y)$ are Fibonacci matrices. More precisely, after recalling some properties of $\mathbf{S}_m(x, y)$ which are valid for arbitrary couples (x, y), we work out closed-form expressions for the entries $s_{h,k}^{(n)}(x, y)$ of $\mathbf{S}_4^n(x, y)$ and prove that, for special values of the above-mentioned couples, they involve Fibonacci numbers. Further, as application examples, some of these expressions are used, jointly with certain matrix expansions, to obtain some hopefully new Fibonacci identities. It is worth mentioning that the existence of some relations between matrices $\mathbf{S}_m(x, y)$ and the Fibonacci numbers is well known (e.g., see [8] and [9]). For example, the corollary to Theorem 1 of [8] tells us that the permanent of $\mathbf{S}_m(1, 1)$ equals F_{m+1}.

2. PRELIMINARY RESULTS

The fundamental properties of $\mathbf{S}_m(x, y)$ have been investigated in [2] where, in particular, the following compact form for the generic entry $f_{h,k}(x, y)$ of any admissible function $f[\mathbf{S}_m(x, y)]$ has been established:

$$f_{h,k}(x,y) = \frac{2}{m+1}\sum_{j=1}^{m} f[\lambda_j(x,y)]\sin\frac{jh\pi}{m+1}\sin\frac{jk\pi}{m+1} \quad (1 \le h, k \le m), \tag{2.1}$$

where (see Theorem D1 of [3])

$$\lambda_j(x,y) = x + 2y\cos\frac{j\pi}{m+1} \quad (j = 1, 2, \ldots, m) \tag{2.2}$$

are the eigenvalues of $\mathbf{S}_m(x, y)$.

Remark 1: It is worth noting that formula (2.1) also works if $\mathbf{S}_m(x, y)$ is replaced by any function $g[\mathbf{S}_m(x, y)]$ which, in general, is not a Toeplitz tridiagonal matrix. It is sufficient to replace $f(x+2y\cos\frac{j\pi}{m+1})$ by $f[g(x+2y\cos\frac{j\pi}{m+1})]$.

Specializing f in (2.1) to the n^{th} power $(n \ge 0)$ yields the desired expression for the entries $s_{h,k}^{(n)}(x, y)$ of $\mathbf{S}_m^n(x, y)$. Namely, we get the relation

$$s_{h,k}^{(n)}(x, y) = \frac{2}{m+1}\sum_{j=1}^{m}\left(x+2y\cos\frac{j\pi}{m+1}\right)^n \sin\frac{jh\pi}{m+1}\sin\frac{jk\pi}{m+1} \quad (1 \le h, k \le m) \tag{2.3}$$

which, in the case $m = 4$ that is of interest to us, becomes

$$s_{h,k}^{(n)}(x, y) = \frac{2}{5}\sum_{j=1}^{4}\left(x+2y\cos\frac{j\pi}{5}\right)^n \sin\frac{jh\pi}{5}\sin\frac{jk\pi}{5} \quad (1 \le h, k \le 4). \tag{2.4}$$

Remark 2: For integral values of x and y $(x+y \ne 0)$, the eigenvalues of $\mathbf{S}_4(x, y)$ are nonzero [see (2.2)]. Therefore, under this condition, (2.4) applies for negative values of n as well.

Formula (2.3) [and, in particular, its specialization (2.4)] will play a crucial role throughout the proofs of the results established in this paper. Since it comes from expression (2.1), which has been established in an unpublished paper, the reader might be interested in an alternative proof of (2.3). It can be obtained by induction on n. To do this, we need the following two trigonometrical identities, the proofs of which are omitted for the sake of brevity:

$$\sum_{j=1}^{m}\sin\frac{jh\pi}{m+1}\sin\frac{jk\pi}{m+1} = \frac{m+1}{2}\delta_{h,k}, \tag{2.5}$$

where $\delta_{h,k} = 1\ (0)$ if $h = (\ne)\ k$ is the Kronecker symbol;

$$\sin(p-q)+\sin(p+q) = 2\sin p\cos q. \tag{2.6}$$

Proof of (2.3) (by induction on n): By virtue of (2.5), expression (2.3) clearly holds for $n=0$. For $n=1$, from identities (2.5) and (2.6) it is not hard to see that (2.3) holds as well. Suppose it holds for a certain $n>1$.

Case 1: $k \ne 1, m$. For the inductive step $n \to n+1$, put $\sin[jr\pi/(m+1)] = s(j, r)$ and $\cos[j\pi/(m+1)] = c(j)$ for notational convenience, and use (1.1) and the inductive hypothesis to write

$$\begin{aligned}
s_{h,k}^{(n+1)} &= xs_{h,k}^{(n)} + y(s_{h,k-1}^{(n)} + s_{h,k+1}^{(n)}) \quad \text{(by the usual matrix multiplication rule)}\\
&= \frac{2}{m+1}\left\{x\sum_{j=1}^{m}\lambda_j^n s(j, h)s(j, k) + y\sum_{j=1}^{m}\lambda_j^n s(j, h)[s(j, k-1)+s(j, k+1)]\right\}\\
&= \frac{2}{m+1}\sum_{j=1}^{m}[x\lambda_j^n s(j, h)s(j, k) + 2y\lambda_j^n s(j, h)s(j, k)c(j)] \quad \text{[from (2.6)]}\\
&= \frac{2}{m+1}\sum_{j=1}^{m}\lambda_j^n[x+2yc(j)]s(j, h)s(j, k)\\
&= \frac{2}{m+1}\sum_{j=1}^{m}\lambda_j^{n+1}s(j, h)s(j, k) \quad \text{[from (2.2)].}
\end{aligned}$$

Case 2: $k = 1$ or m. The proof of this case can be carried out by means of arguments similar to (but much simpler than) those of Case 1, and is omitted for brevity. Q.E.D.

Observe that, since $s_{h,k}^{(1)}(x, y) = x\delta_{h,k} + y\delta_{|h-k|,1}$, letting $n = 1$ in (2.3) and using (2.5) yields the noteworthy trigonometrical identity

$$\sum_{j=1}^{m} \cos\frac{j\pi}{m+1}\sin\frac{jh\pi}{m+1}\sin\frac{jk\pi}{m+1} = \frac{m+1}{4}\delta_{|h-k|,1}. \tag{2.7}$$

3. SOME FIBONACCI MATRICES

In this section, some couples (x, y) for which $\mathbf{S}_4(x, y)$ is a Fibonacci matrix are shown, and closed-form expressions for the entries $s_{h,k}^{(n)}(x, y)$ of $\mathbf{S}_4^n(x, y)$ are established. Of course, since $s_{h,k}^{(n)}(px, py) = p^n s_{h,k}^{(n)}(x, y)$, only *coprime* values of x and y are considered. Further, since it can be proved that the above entries enjoy the symmetry properties

$$\begin{aligned} &s_{11} = s_{44}, \\ &s_{12} = s_{21} = s_{34} = s_{43}, \\ &s_{13} = s_{31} = s_{24} = s_{42}, \\ &s_{14} = s_{41}, \\ &s_{22} = s_{33}, \\ &s_{23} = s_{32}, \end{aligned} \tag{3.1}$$

[here, $s_{h,k}$ stands for $s_{h,k}^{(n)}(x, y)$], expressions will be given only for $s_{1,k}$ $(1 \le k > 4)$, s_{22}, and s_{23}. For the sake of brevity, only a few among these results will be proved in detail by using relation (2.4). On the other hand, once the results have been established, induction on n and some usual Fibonacci identities may provide alternative (even though more tedious) proofs for all of them.

3.1 The Matrix $\mathbf{S}_4(0, 1)$

$$s_{11}^{(n)}(0, 1) = F_{n-1}[1 + (-1)^n]/2, \tag{3.2}$$

$$s_{12}^{(n)}(0, 1) = F_n[1 - (-1)^n]/2, \tag{3.3}$$

$$s_{13}^{(n)}(0, 1) = F_n[1 + (-1)^n]/2, \tag{3.4}$$

$$s_{14}^{(n)}(0, 1) = F_{n-1}[1 - (-1)^n]/2, \tag{3.5}$$

$$s_{22}^{(n)}(0, 1) = F_{n+1}[1 + (-1)^n]/2, \tag{3.6}$$

$$s_{23}^{(n)}(0, 1) = F_{n+1}[1 - (-1)^n]/2. \tag{3.7}$$

From (3.2)-(3.7), it is immediately seen that

$$\begin{cases} s_{11}^{(n)}(0, 1) + s_{14}^{(n)}(0, 1) = F_{n-1}, \\ s_{12}^{(n)}(0, 1) + s_{13}^{(n)}(0, 1) = F_n, \\ s_{22}^{(n)}(0, 1) + s_{23}^{(n)}(0, 1) = F_{n+1}, \end{cases} \tag{3.8}$$

and

$$\mathrm{Tr}[\mathbf{S}_4^n(0, 1)] = L_n[1 + (-1)^n] \quad \text{[from (3.1)]}, \tag{3.9}$$

where the trace Tr(**M**) of any square matrix **M** is the sum of its diagonal entries (or that of its eigenvalues).

Proof of (3.2): From (2.4), write

$$s_{11}^{(n)}(0,1)=\frac{2}{5}\sum_{j=1}^{4}\left(2\cos\frac{j\pi}{5}\right)^n\sin^2\frac{j\pi}{5}, \tag{3.10}$$

and observe that $2\cos(j\pi/5)=\alpha,-\beta,\beta$, and $-\alpha$ for $j=1,2,3$, and 4, respectively, where $\alpha=1-\beta=(1+\sqrt{5})/2$. Moreover, observe that $\sin^2(j\pi/5)=(1+\beta^2)/4$ (for $j=1$ and 4) and $(1+\alpha^2)/4$ (for $j=2$ and 3), so that, by using the Binet forms for Fibonacci and Lucas numbers and the property $\alpha\beta=-1$, (3.10) can be rewritten as

$$\begin{aligned}s_{11}^{(n)}(0,1)&=\frac{1}{10}[\alpha^n+\alpha^{n-2}+(-1)^n(\beta^n+\beta^{n-2})+\beta^n+\beta^{n-2}+(-1)^n(\alpha^n+\alpha^{n-2})]\\&=\begin{cases}(L_n+L_{n-2})/5=F_{n-1} & (n\text{ even}),\\ 0 & (n\text{ odd}).\end{cases}\end{aligned}$$

The proofs of (3.3)-(3.7) can be carried out by means of analogous arguments.

3.2 The Matrix $S_4(1,1)$

$$s_{11}^{(n)}(1,1)=(F_{2n-1}+F_{n+1})/2, \tag{3.11}$$
$$s_{12}^{(n)}(1,1)=(F_{2n}+F_n)/2, \tag{3.12}$$
$$s_{13}^{(n)}(1,1)=(F_{2n}-F_n)/2, \tag{3.13}$$
$$s_{14}^{(n)}(1,1)=(F_{2n-1}-F_{n+1})/2, \tag{3.14}$$
$$s_{22}^{(n)}(1,1)=(F_{2n+1}+F_{n-1})/2, \tag{3.15}$$
$$s_{23}^{(n)}(1,1)=(F_{2n+1}-F_{n-1})/2. \tag{3.16}$$

From (3.11)-(3.16), it is immediately seen that

$$\begin{cases}s_{11}^{(n)}(1,1)+s_{14}^{(n)}(1,1)=F_{2n-1},\\ s_{12}^{(n)}(1,1)+s_{13}^{(n)}(1,1)=F_{2n},\\ s_{22}^{(n)}(1,1)+s_{23}^{(n)}(1,1)=F_{2n+1},\end{cases} \tag{3.17}$$

and

$$\mathrm{Tr}[\mathbf{S}_4^n(1,1)]=L_n+L_{2n}\quad\text{[from (3.1)]}. \tag{3.18}$$

Proof of (3.14): From (2.4), write

$$s_{14}^{(n)}(1,1)=\frac{2}{5}\sum_{j=1}^{4}\left(1+2\cos\frac{j\pi}{5}\right)^n\sin\frac{j\pi}{5}\sin\frac{4j\pi}{5}, \tag{3.19}$$

and observe that $1+2\cos(j\pi/5)=\alpha^2,\alpha,\beta^2$, and β for $j=1,2,3$, and 4, respectively. Moreover, observe that $\sin(j\pi/5)\sin(4j\pi/5)=(\beta^2+1)/4$, $-(\alpha^2+1)/4$, $(\alpha^2+1)/4$, and $-(\beta^2+1)/4$ for $j=1,2,3$, and 4, respectively. Consequently, by using the Binet form for Lucas numbers and the property $\alpha\beta=-1$, (3.19) can be rewritten as

$$s_{14}^{(n)}(1,1)=\frac{1}{10}(\alpha^{2n-2}+\alpha^{2n}-\alpha^{n+2}-\alpha^n+\beta^{2n-2}+\beta^{2n}-\beta^{n+2}-\beta^n)=$$

$$= \frac{1}{10}(L_{2n-2} + L_{2n} - L_{n+2} - L_n)$$
$$= \frac{1}{10}(5F_{2n-1} + 5F_{n+1}) = (F_{2n-1} + F_{n+1}) / 2.$$

The proofs of (3.11)-(3.13) and (3.15)-(3.16) can be carried out by means of analogous arguments.

3.3 The Matrix $\mathbf{S}_4(1, 2)$

$$s_{11}^{(n)}(1,2) = (F_{3n-1} + 5^{\lfloor n/2 \rfloor}) / 2, \tag{3.20}$$
$$s_{12}^{(n)}(1,2) = \{F_{3n} + 5^{(n-1)/2}[1-(-1)^n]\} / 2, \tag{3.21}$$
$$s_{13}^{(n)}(1,2) = \{F_{3n} - 5^{(n-1)/2}[1-(-1)^n]\} / 2, \tag{3.22}$$
$$s_{14}^{(n)}(1,2) = (F_{3n-1} - 5^{\lfloor n/2 \rfloor}) / 2, \tag{3.23}$$
$$s_{22}^{(n)}(1,2) = (F_{3n+1} + (-1)^n 5^{\lfloor n/2 \rfloor}) / 2, \tag{3.24}$$
$$s_{23}^{(n)}(1,2) = (F_{3n+1} - (-1)^n 5^{\lfloor n/2 \rfloor}) / 2, \tag{3.25}$$

where the symbol $\lfloor \cdot \rfloor$ denotes the greatest integer function.

From (3.20)-(3.25), it is immediately seen that

$$\begin{cases} s_{11}^{(n)}(1,2) + s_{14}^{(n)}(1,2) = F_{3n-1}, \\ s_{12}^{(n)}(1,2) + s_{13}^{(n)}(1,2) = F_{3n}, \\ s_{22}^{(n)}(1,2) + s_{23}^{(n)}(1,2) = F_{3n+1}, \end{cases} \tag{3.26}$$

and

$$\mathrm{Tr}[\mathbf{S}_4^n(1,2)] = L_{3n} + 5^{n/2}[1+(-1)^n] \quad \text{[from (3.1)]}. \tag{3.27}$$

As an example application of Remark 2, we provide the expression for the entry $s_{11}^{(-n)}(1,2)$ of $\mathbf{S}_4^{-n}(1,2)$. Namely, we have

$$s_{11}^{(-n)}(1,2) = [(-1)^n F_{3n+1} + 5^{-\lfloor (n+1)/2 \rfloor}] / 2. \tag{3.28}$$

The proof of (3.28) is left as an exercise for the interested reader.

Proof of (3.21): From (2.4), write

$$s_{12}^{(n)}(1,2) = \frac{2}{5}\sum_{j=1}^{4}\left(1 + 4\cos\frac{j\pi}{5}\right)^n \sin\frac{j\pi}{5}\sin\frac{2j\pi}{5}, \tag{3.29}$$

and observe that $1 + 4\cos(j\pi/5) = \alpha^3, \sqrt{5}, \beta^3$, and $-\sqrt{5}$ for $j = 1, 2, 3$, and 4, respectively. Moreover, observe that $\sin(j\pi/5)\sin(2j\pi/5) = \sqrt{5}/4$ (for $j = 1$ and 2) and $-\sqrt{5}/4$ (for $j = 3$ and 4), so that, by using the Binet form for Fibonacci numbers, (3.29) can be rewritten as

$$s_{12}^{(n)}(1,2) = \frac{1}{10}\{\sqrt{5}(\alpha^{3n} - \beta^{3n}) + \sqrt{5}(\sqrt{5})^n[1-(-1)^n]\}$$
$$= \begin{cases} F_{3n}/2 & (n \text{ even}), \\ F_{3n}/2 + 5^{(n-1)/2} & (n \text{ odd}). \end{cases}$$

The proofs of (3.20) and (3.22)-(3.25) can be carried out by means of analogous arguments.

3.4 The Matrix $\mathbf{S}_4(F_s, F_{s+1})$

The expressions (3.8), (3.17), and (3.26) can be generalized as follows.

Proposition 1: If s is any integer, then

$$\begin{cases} s_{11}^{(n)}(F_s, F_{s+1}) + s_{14}^{(n)}(F_s, F_{s+1}) = F_{(s+1)n-1}, \\ s_{12}^{(n)}(F_s, F_{s+1}) + s_{13}^{(n)}(F_s, F_{s+1}) = F_{(s+1)n}, \\ s_{22}^{(n)}(F_s, F_{s+1}) + s_{23}^{(n)}(F_s, F_{s+1}) = F_{(s+1)n+1}. \end{cases} \tag{3.30}$$

Proof: To save space, we shall confine ourselves to proving only the second identity of (3.30). The proofs of the first and third identities can be obtained in a similar manner.

First, from (2.4), write

$$s_{1,k}^{(n)}(F_s, F_{s+1}) = \frac{2}{5}\sum_{j=1}^{4}\left(F_s + 2F_{s+1}\cos\frac{j\pi}{5}\right)^n \sin\frac{j\pi}{5}\sin\frac{kj\pi}{5}. \tag{3.31}$$

Then, write down the following chart:

j	$\left(F_s + 2F_{s+1}\cos\frac{j\pi}{5}\right)^n$	$\sin\frac{j\pi}{5}\sin\frac{2j\pi}{5}$	$\sin\frac{j\pi}{5}\sin\frac{3j\pi}{5}$
1	$(F_s + \alpha F_{s+1})^n = \alpha^{(s+1)n}$	$\sqrt{5}/4$	$\sqrt{5}/4$
2	$(F_s - \beta F_{s+1})^n$	$\sqrt{5}/4$	$-\sqrt{5}/4$
3	$(F_s + \beta F_{s+1})^n = \beta^{(s+1)n}$	$-\sqrt{5}/4$	$-\sqrt{5}/4$
4	$(F_s - \alpha F_{s+1})^n$	$-\sqrt{5}/4$	$\sqrt{5}/4$

Finally, put $k = 1$ and 2 in (3.31) and use the chart to obtain

$$\begin{aligned} & s_{12}^{(n)}(F_s, F_{s+1}) + s_{13}^{(n)}(F_s, F_{s+1}) \\ & = \frac{1}{2\sqrt{5}}[\alpha^{(s+1)n} - \beta^{(s+1)n} + (F_s - \beta F_{s+1})^n - (F_s - \alpha F_{s+1})^n] \\ & + \frac{1}{2\sqrt{5}}[\alpha^{(s+1)n} - \beta^{(s+1)n} - (F_s - \beta F_{s+1})^n + (F_s - \alpha F_{s+1})^n] = F_{(s+1)n}. \end{aligned}$$

3.5 Miscellany

The identities established in Subsections 3.1-3.4 represent only a small sample of the possibilities available to us. As a minor example, we leave the proofs of the following results to the interested reader:

$$s_{22}^{(n)}(2, 1) = (5^{\lfloor n/2 \rfloor} W_{n+1} + F_{2n-1})/2, \tag{3.32}$$

where W stands for F (n even) or L (n odd);

$$s_{h,k}^{(n)}(1, -1) = (-1)^{h+k} s_{h,k}^{(n)}(1, 1); \tag{3.33}$$

$$\mathrm{Tr}[\mathbf{S}_4^n(x, 1)] = 2\sum_{j=0}^{\lfloor n/2 \rfloor}\binom{n}{2j}x^{n-2j}L_{2j} \tag{3.34}$$

which, for $x = 0$, reduces to (3.9) by virtue of the usual assumption $0^z = \delta_{z,0}$ $(z \geq 0)$.

4. APPLICATION EXAMPLES

In this section some Fibonacci matrices $\mathbf{S}_4(x, y)$ are used jointly with certain matrix expansions to get Fibonacci and combinatorial relations. Their novelty may be questioned; nevertheless, our aim here is to illustrate some ways of using the results presented in Section 3.

Example 1: Consider the matrix inverse expansion (see [5, p. 113])

$$(\mathbf{I}-\mathbf{A})^{-1} = \sum_{n=0}^{\infty} \mathbf{A}^n \quad (|\lambda|<1;\ \lambda,\text{ any eigenvalue of }\mathbf{A}), \tag{4.1}$$

and put $\mathbf{A} = \frac{1}{2}\mathbf{S}_4(0,1)$ $[$whence $\mathbf{I}_4 - \mathbf{A} = \frac{1}{2}\mathbf{S}_4(2,-1)]$ in (4.1), thus getting the relation

$$\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\mathbf{S}_4^n(0,1) = \mathbf{S}_4^{-1}(2,-1). \tag{4.2}$$

Then, from (2.4) and Remark 2, write

$$\begin{aligned} s_{12}^{(-1)}(2,-1) &= \frac{2}{5}\sum_{j=1}^{4}\left(2-2\cos\frac{j\pi}{5}\right)^{-1}\sin\frac{j\pi}{5}\sin\frac{2j\pi}{5} \\ &= \frac{1}{2\sqrt{5}}[\beta^{-2}+(\beta^2+1)^{-1}-\alpha^{-2}-(\alpha^2+1)^{-1}] = \frac{3}{5}. \end{aligned} \tag{4.3}$$

Finally, use (4.2), (3.3), and (4.3) to obtain the relation

$$\sum_{n=0}^{\infty}\frac{1}{2^{2n+1}}F_{2n+1} = \frac{6}{5}. \tag{4.4}$$

Example 2: Consider the matrix logarithm expansion (see [5, p. 113])

$$\ln\mathbf{A} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(\mathbf{A}-\mathbf{I})^n}{n} \quad (|\lambda-1|<1;\ \lambda,\text{ any eigenvalue of }\mathbf{A}), \tag{4.5}$$

and put $\mathbf{A} = \mathbf{S}_4^2(0, 1/\sqrt{2})$ in (4.5), thus getting the relation

$$\ln\mathbf{S}_4^2(0,1/\sqrt{2}) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}[\mathbf{S}_4^2(0,1/\sqrt{2})-\mathbf{I}_4]^n. \tag{4.6}$$

First, find the upper-left entry l_{11} of the matrix on the left-hand side of (4.6). The eigenvalues μ_j of $\mathbf{S}_4(0,1/\sqrt{2})$ are $\mu_1=\mu_4=\alpha/\sqrt{2}$ and $\mu_2=\mu_3=\beta/\sqrt{2}$, so that, from (2.3) and Remark 1, we get

$$\begin{aligned} l_{11} &= \frac{2}{5}\sum_{j=1}^{4}\ln\mu_j^2\sin^2\frac{j\pi}{5} = \frac{2}{5}\left[\frac{1+\beta^2}{2}\ln\frac{\alpha^2}{2}+\frac{1+\alpha^2}{2}\ln\frac{\beta^2}{2}\right] \\ &= \frac{1}{5}[(1+\beta^2)\ln\alpha^2+(1+\alpha^2)\ln\beta^2-5\ln 2] = \frac{1}{5}[\beta^2\ln\alpha^2+\alpha^2\ln\beta^2-5\ln 2] \\ &= \frac{1}{5}[(\beta^2-\alpha^2)\ln\alpha^2-5\ln 2] = -\frac{2}{\sqrt{5}}\ln\alpha-\ln 2. \end{aligned} \tag{4.7}$$

Then, find the upper-left entry $t_{11}^{(n)}$ of $[\mathbf{S}_4^2(0,1/\sqrt{2})-\mathbf{I}_4]^n$. The eigenvalues ξ_j of $\mathbf{S}_4^2(0,1/\sqrt{2})$ $-\,\mathbf{I}_4$ are $\xi_1=\xi_4=\mu_1^2-1$ and $\xi_2=\xi_3=\mu_2^2-1$, so that, from (2.4) and Remark 1, we can write

$$
\begin{aligned}
t_{11}^{(n)} &= \frac{2}{5}\sum_{j=1}^{4}\xi_j^n \sin^2\frac{j\pi}{5} = \frac{1}{5}\left[\left(\frac{\alpha^2}{2}-1\right)^n(1+\beta^2)+\left(\frac{\beta^2}{2}-1\right)^n(1+\alpha^2)\right] \\
&= \frac{1}{5}\left[\left(-\frac{\beta}{2}\right)^n(1+\beta^2)+\left(-\frac{\alpha}{2}\right)^n(1+\alpha^2)\right] \\
&= \frac{(-1)^n}{2^n 5}(L_n+L_{n-2}) = \frac{(-1)^n}{2^n}F_{n+1}.
\end{aligned} \tag{4.8}
$$

Finally, use (4.6)-(4.8) to obtain the relation

$$\sum_{n=1}^{\infty}\frac{F_{n+1}}{n2^n} = \frac{2}{\sqrt{5}}\ln\alpha + \ln 2. \tag{4.9}$$

Example 3: Since $\mathbf{S}_4(2,1) = \mathbf{S}_4(0,1)+\mathbf{S}_4(2,0)$, from (1.4) of [4] write

$$\mathbf{S}_4^n(2,1) = \sum_{j=0}^{n}\binom{n}{j}\mathbf{S}_4^j(0,1)\mathbf{S}_4^{n-j}(2,0), \tag{4.10}$$

whence, from (3.6) and (3.32), one obtains the relation

$$\sum_{j=0}^{n}\binom{n}{j}F_{j+1}\frac{1+(-1)^j}{2}2^{n-j} = (5^{\lfloor n/2\rfloor}W_{n+1}+F_{2n-1})/2$$

[where W stands for F (n even) or L (n odd)], which can be rewritten as (cf. identities I_{41} and I_{42} of [6])

$$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}2^{n-2j+1}F_{2j+1} = 5^{\lfloor n/2\rfloor}W_{n+1}+F_{2n-1}. \tag{4.11}$$

Example 4: Put $\mathbf{A}=\mathbf{B}=\mathbf{S}_4(0,1)$ in (1.5) of [4], and write

$$2\mathbf{S}_4^n(0,1) = \sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\frac{n}{n-j}\binom{n-j}{j}\mathbf{S}_4^{2j}(0,1)\mathbf{S}_4^{n-2j}(0,2) \quad (n\ge 1). \tag{4.12}$$

First, observe that the upper-left entry of the matrix on the left-hand side of (4.12) is

$$2s_{11}^{(n)}(0,1) = F_{n-1}[1+(-1)^n] \quad \text{[from (3.2)]}. \tag{4.13}$$

Then, use (3.2)-(3.5), (3.1), and the usual matrix multiplication rules to write the upper-left entry u_{11} of $\mathbf{S}_4^{2j}(0,1)\mathbf{S}_4^{n-2j}(0,2)$ as

$$
\begin{aligned}
u_{11} &= 2^{n-2j-1}[1+(-1)^n](F_{2j-1}F_{n-2j-1}+F_{2j}F_{n-2j}) \\
&= 2^{n-2j-1}F_{n-1}[1+(-1)^n] \quad \text{(from identity } I_{26} \text{ of [6])}.
\end{aligned} \tag{4.14}
$$

Finally, use (4.12)-(4.14) to obtain the combinatorial identity

$$\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\frac{n}{n-j}\binom{n-j}{j}2^{n-2j-1} = 1 \quad (n\ge 1). \tag{4.15}$$

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REFERENCES

1. M. Bicknell & V. E. Hoggatt, Jr., eds. *A Primer for the Fibonacci Numbers*. Santa Clara, CA: The Fibonacci Association, 1973.
2. O. Brugia & P. Filipponi. "Functions of Tridiagonal Symmetric Toeplitz Matrices." Int. Rept. 3T03195. Rome: Fondazione Ugo Bordoni.
3. H. R. P. Ferguson. "The Fibonacci Pseudogroup, Characteristic Polynomials and Eigenvalues of Tridiagonal Matrices, Periodic Linear Recurrence Systems and Application to Quantum Mechanics." *The Fibonacci Quarterly* **16.5** (1978):435-47.
4. P. Filipponi. "Waring's Formula, the Binomial Formula, and Generalized Fibonacci Matrices." *The Fibonacci Quarterly* **30.3** (1992):225-31.
5. F. R. Gantmacher. *The Theory of Matrices*. Vol. 1. New York: Chelsea, 1977.
6. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969; rpt. Santa Clara, CA: The Fibonacci Association, 1979.
7. C. H. King. "Some Properties of the Fibonacci Numbers." Master's Thesis, San Jose State College, June 1960.
8. G.-Y. Lee & S.-G. Lee. "A Note on Generalized Fibonacci Numbers." *The Fibonacci Quarterly* **33.3** (1995):273-78.
9. D. H. Lehmer. "Fibonacci and Related Sequences in Periodic Tridiagonal Matrices." *The Fibonacci Quarterly* **13.2** (1975):150-58.

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