ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stan@wwa.com *on the Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-848 ***Proposed by Russell Euler's Fall 1997 Number Theory Class, Northwest Missouri State University, Maryville, MO***

Prove that $F_nF_{n+1} - F_{n+6}F_{n-5} = 40(-1)^{n+1}$ for all integers n.

B-849 ***Proposed by Larry Zimmerman & Gilbert Kessler, New York, NY***

If F_a, F_b, F_c, x forms an increasing arithmetic progression, show that x must be a Lucas number.

B-850 ***Proposed by Al Dorp, Edgemere, NY***

Find distinct positive integers a, b, and c so that $F_n = 17F_{n-a} + cF_{n-b}$ is an identity.

B-851 ***Proposed by Pentti Haukkanen, University of Tampere, Tampere, Finland***

Consider the repeating sequence $\langle A_n \rangle_{n=0}^{\infty} = 0, 1, -1, 0, 1, -1, 0, 1, -1, \ldots$.

(a) Find a recurrence formula for A_n.

(b) Find an explicit formula for A_n of the form $(a^n - b^n)/(a-b)$.

B-852 ***Proposed by Stanley Rabinowitz, Westford, MA***

Evaluate

$$\begin{vmatrix} F_0 & F_1 & F_2 & F_3 & F_4 \\ F_9 & F_8 & F_7 & F_6 & F_5 \\ F_{10} & F_{11} & F_{12} & F_{13} & F_{14} \\ F_{19} & F_{18} & F_{17} & F_{16} & F_{15} \\ F_{20} & F_{21} & F_{22} & F_{23} & F_{24} \end{vmatrix}.$$

B-853 ***Proposed by Gene Ward Smith, Brunswick, ME***

Consider the recurrence $f(n+1)=n(f(n)+f(n-1))$ with initial conditions $f(0)=1$ and $f(1)=0$. Find a closed form for the sum

$$\sum_{k=0}^{n}\binom{n}{k}f(k).$$

NOTE: The Elementary Problems Column is in need of more **easy**, yet elegant and nonroutine problems.

SOLUTIONS

Pattern Detective

B-832 ***Proposed by Andrew Cusumano, Great Neck, NY***
(Vol. 35, no. 3, August 1997)

Find a pattern in the following numerical identities and create a formula expressing a more general result.

$$\begin{aligned}
3^5+2^5+1^5+1^5 &= 5\cdot 3^4-128\\
5^5+3^5+2^5+1^5+1^5 &= 8\cdot 5^4-128-2\cdot3\cdot5(19+2\cdot3\cdot5)\\
8^5+5^5+3^5+2^5+1^5+1^5 &= 13\cdot 8^4-128-2\cdot3\cdot5(19+2\cdot3\cdot5)\\
&\quad -3\cdot5\cdot8(19+2\cdot3\cdot5+2\cdot5\cdot8)\\
13^5+8^5+5^5+3^5+2^5+1^5+1^5 &= 21\cdot 13^4-128-2\cdot3\cdot5(19+2\cdot3\cdot5)\\
&\quad -3\cdot5\cdot8(19+2\cdot3\cdot5+2\cdot5\cdot8)\\
&\quad -5\cdot8\cdot13(19+2\cdot3\cdot5+2\cdot5\cdot8+2\cdot8\cdot13)\\
21^5+13^5+8^5+5^5+3^5+2^5+1^5+1^5 &= 34\cdot 21^4-128-2\cdot3\cdot5(19+2\cdot3\cdot5)\\
&\quad -3\cdot5\cdot8(19+2\cdot3\cdot5+2\cdot5\cdot8)\\
&\quad -5\cdot8\cdot13(19+2\cdot3\cdot5+2\cdot5\cdot8+2\cdot8\cdot13)\\
&\quad -8\cdot13\cdot21(19+2\cdot3\cdot5+2\cdot5\cdot8+2\cdot8\cdot13+2\cdot13\cdot21)
\end{aligned}$$

Solution by H.-J. Seiffert, Berlin, Germany

Let k be an integer. Then $F_{k-2}(F_k^2+F_{k-1}F_{k+1})=F_{k-2}(F_k^2+F_{k-1}F_k+F_{k-1}^2)=(F_k-F_{k-1})(F_k^2+F_{k-1}F_k+F_{k-1}^2)=F_k^3-F_{k-1}^3$. Using $F_{k-1}F_{k+1}-F_k^2=(-1)^k$, which is identity ($I_{13}$) of [1], we have

$$F_{k-2}(2F_k^2+(-1)^k)=F_k^3-F_{k-1}^3.$$

Multiplying this equation by $F_{k-1}F_k$, adding F_k^5 on both sides of the resulting equation, and using $F_k^5+F_{k-1}F_k^4=F_{k+1}F_k^4$, we obtain

$$F_k^5+F_{k-2}F_{k-1}F_k(2F_k^2+(-1)^k)=F_{k+1}F_k^4-F_kF_{k-1}^4.$$

Summing as k ranges from 1 to n yields

$$\sum_{k=1}^{n}F_k^5=F_{n+1}F_n^4-\sum_{k=1}^{n}F_{k-2}F_{k-1}F_k(2F_k^2+(-1)^k). \tag{1}$$

From identity (I_{13}), again, we find

$$F_j^2-F_{j-1}^2+(-1)^j=F_{j-1}F_{j+1}-F_{j-1}^2=F_{j-1}F_j.$$

Summing as j ranges from 1 to k gives

$$F_k^2+\frac{1}{2}((-1)^k-1)=\sum_{j=1}^{k}F_{j-1}F_j \quad \text{or} \quad 2F_k^2+(-1)^k=1+2\sum_{j=1}^{k}F_{j-1}F_j.$$

Now equation (1) may be rewritten as

$$\sum_{k=1}^{n}F_k^5=F_{n+1}F_n^4-\sum_{k=1}^{n}F_{k-2}F_{k-1}F_k\left(1+2\sum_{j=1}^{k}F_{j-1}F_j\right).$$

Let $n\geq 4$. Using

$$\sum_{k=1}^{4}F_{k-2}F_{k-1}F_k\left(1+2\sum_{j=1}^{k}F_{j-1}F_j\right)=128 \quad \text{and} \quad 1+2\sum_{j=1}^{4}F_{j-1}F_j=19,$$

we find

$$\sum_{k=1}^{n}F_k^5=F_{n+1}F_n^4-128-\sum_{k=5}^{n}F_{k-2}F_{k-1}F_k\left(19+2\sum_{j=5}^{k}F_{j-1}F_j\right),$$

valid for all $n\geq 4$. For $n=4,5,6,7,$ and 8, this produces the numerical identities given in the proposal.

The proposer sent along many related identities.

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Santa Clara, Calif.: The Fibonacci Association, 1979.

Also solved by Paul S. Bruckman, L. A. G. Dresel, and the proposer.

Newton Meets Lucas

B-833 ***Proposed by Al Dorp, Edgemere, NY***
(Vol. 35, no. 3, August 1997)

For n a positive integer, let $f(x)$ be the polynomial of degree $n-1$ such that $f(k)=L_k$ for $k=1,2,3,\ldots,n$. Find $f(n+1)$.

Solution by Paul S. Bruckman, Highwood, IL

We employ the Pochhammer notation $x^{(m)}=x(x-1)(x-2)\cdots(x-m+1)$ and use Newton's Forward Difference Formula ([1], p. 29), which says that if f_k is a polynomial of degree n, then

$$f_k=f_0+\frac{\Delta f_0}{1!}k^{(1)}+\frac{\Delta^2 f_0}{2!}k^{(2)}+\cdots+\frac{\Delta^n f_0}{n!}k^{(n)},$$

where the operator Δ is defined by $\Delta f(x)=f(x+1)-f(x)$.

In our example, $\Delta f(k)=f(k+1)-f(k)=L_{k+1}-L_k=L_{k-1}$ for $k=1,2,\ldots,n-1$. Similarly, $\Delta^2 f(k)=\Delta(\Delta f(k))=\Delta L_{k-1}=L_{k-2}$ for $k=1,2,\ldots,n-2$. Continuing, we find $\Delta^3 f(k)=L_{k-3}$ for $k=1,2,\ldots,n-3$, etc., until $\Delta^{n-1}f(1)=L_{2-n}$.

Applying Newton's formula, we obtain

$$f(x+1)=\sum_{k=0}^{n-1}\frac{L_{1-k}}{k!}x^{(k)}=\sum_{k=0}^{n-1}(-1)^{k-1}L_{k-1}x^{(k)}/k!.$$

Then

$$\begin{aligned} f(n+1) &= \sum_{k=0}^{n-1}(-1)^{k-1}\binom{n}{k}L_{k-1} \\ &= \sum_{k=1}^{n-1}\binom{n}{k}[(-\alpha)^{k-1}+(-\beta)^{k-1}] \\ &= \beta(1-\alpha)^n+\alpha(1-\beta)^n-(-\alpha)^{n-1}-(-\beta)^{n-1} \\ &= \beta^{n+1}+\alpha^{n+1}+(-1)^n[\alpha^{n-1}+\beta^{n-1}] \\ &= L_{n+1}+(-1)^nL_{n-1}. \end{aligned}$$

This is equivalent to L_n if n is odd and $5F_n$ if n is even.

Reference

1. Ronald E. Mickens. *Difference Equations*. New York: Van Nostrand Reinhold, 1990.

Also solved by Charles K. Cook, L. A. G. Dresel, Hans Kappus, Harris Kwong, R. Horace McNutt, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Radical Inequality

B-834 ***Proposed by Zdravko F. Starc, Vrsac, Yugoslavia (Vol. 35, no. 3, August 1997)***

For x a real number and n an integer larger than 1, prove that

$$(x+1)F_1+(x+2)F_2+\cdots+(x+n)F_n<2^n\sqrt{\frac{n(n+1)(2n+1+6x)+6nx^2}{6}}.$$

Solution by the proposer

From the identity $\alpha^n=\alpha F_n+F_{n-1}$, we see that for $n\geq 2$,

$$F_n=\alpha^{n-1}-\frac{F_{n-1}}{\alpha}<\alpha^{n-1}<2^{n-1}.$$

Thus,

$$\sqrt{F_nF_{n+1}}<2^n. \qquad (*)$$

Cauchy's Inequality ([1], p. 20) says that for all real numbers a_i and b_i,

$$(a_1b_1+a_2b_2+\cdots+a_nb_n)^2\leq(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2).$$

Let $a_i=x+i$ and $b_i=F_i$. Then, using the facts

$$F_1^2+F_2^2+\cdots+F_n^2=F_nF_{n+1},$$

$$(x+1)^2+(x+2)^2+\cdots+(x+n)^2=\frac{n(n+1)(2n+1+6x)+6nx^2}{6},$$

and the inequality $(*)$, we obtain the desired inequality for all x for which the radicand is non-negative.

Reference

1. D. S. Mitrinovic. *Analytic Inequalities*. Berlin: Springer Verlag, 1970.

Also solved by Paul S. Bruckman.

Cryptarithmic Identity

B-836 ***Proposed by Al Dorp, Edgemere, NY***
(Vol. 35, no. 4, November 1997)

Replace each of W, X, Y, and Z by either F or L to make the following an identity:

$$W_n^2 - 6X_{n+1}^2 + 2Y_{n+2}^2 - 3Z_{n+3}^2 = 0.$$

Solution by L. A. G. Dresel, Reading, England

Putting $n = -1$, we find

$$W_1^2 + 2Y_1^2 = 3(Z_2^2 + 2X_0^2). \tag{a}$$

Putting $n = 0$, we find

$$W_0^2 + 2Y_2^2 = 3(Z_3^2 + 2X_1^2). \tag{b}$$

Since $F_1 = L_1 = 1$, we have $W_1 = Y_1 = 1$, so that (a) implies $3 = 3(Z_2^2 + 2X_0^2)$, giving $X_0 = F_0 = 0$ and $Z_2 = F_2 = 1$. Substituting in (b) gives $W_0^2 + 2Y_2^2 = 3(F_3^2 + 2F_1^2) = 18$, which requires $W_0 = F_0 = 0$ and $Y_2 = L_2 = 3$. Therefore, the required identity is

$$F_n^2 - 6F_{n+1}^2 + 2L_{n+2}^2 - 3F_{n+3}^2 = 0.$$

This is satisfied for $n = -1$, $n = 0$, and also for $n = 1$. Therefore, by the Verification Theorem of [1], this is an identity for all values of n.

Reference

1. L. A. G. Dresel. "Transformations of Fibonacci-Lucas Identities." In *Applications of Fibonacci Numbers* **5:**169-84. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1993.

Also solved by Paul S. Bruckman, Charles K. Cook, Russell Jay Hendel, Daina A. Krigens, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Polynomial Remainder

B-837 ***Proposed by Joseph J. Koštál, Chicago, IL***
(Vol. 35, no. 4, November 1997)

Let $P(x) = x^{1997} + x^{1996} + x^{1995} + \cdots + x^2 + x + 1$ and let $R(x)$ be the remainder when $P(x)$ is divided by $x^2 - x - 1$. Show that $R(x)$ is divisible by L_{999}.

Solution by L. A. G. Dresel, Reading, England

Consider, more generally, the polynomial

$$P_n(x) = x^{2n-1} + x^{2n-2} + x^{2n-3} + \cdots + x^2 + x + 1 = \frac{x^{2n} - 1}{x - 1}.$$

Then, if $R(x)$ is the remainder on dividing by $x^2 - x - 1$, we have the identity

$$P_n(x) = (x^2 - x - 1)Q(x) + R(x),$$

where $Q(x)$ is a polynomial in x. Putting $x = \alpha$, and using the fact that $\alpha^2 - \alpha - 1 = 0$, we find that $P_n(\alpha) = R(\alpha)$ and $P_n(\beta) = R(\beta)$. Hence, $\alpha^{2n} - 1 = (\alpha - 1)R(\alpha)$ and $\beta^{2n} - 1 = (\beta - 1)R(\beta)$. Now $\alpha - 1 = -\beta$, $\beta - 1 = -\alpha$, and $R(x) = Ax + B$, where A and B are constants. Hence, $\alpha^{2n} - 1 =$

$A - B\beta$ and $\beta^{2n} - 1 = A - B\alpha$. Subtracting gives $(\alpha-\beta)F_{2n} = B(\alpha-\beta)$, so that $B = F_{2n} = F_n L_n$, while adding gives $L_{2n} - 2 = 2A - B$. When n is odd, we have $L_n^2 = (\alpha^n + \beta^n)^2 = L_{2n} - 2 = 2A - B$, whereas when n is even, we have $5F_n^2 = (\alpha^n - \beta^n)^2 = L_{2n} - 2 = 2A - B$. It follows that when n is odd, $R(x)$ is divisible by L_n, whereas when n is even, $R(x)$ is divisible by F_n.

Also solved by Charles Ashbacher, David M. Bloom, Paul S. Bruckman, Al Dorp, Russell Euler & Jawad Sadek, Russell Jay Hendel, Hans Kappus, Harris Kwong, R. Horace McNutt, Bob Prielipp, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Composite Linear Recurrence

B-838 ***Proposed by Peter G. Anderson, Rochester Institute of Technology, Rochester, NY (Vol. 35, no. 4, November 1997)***

Define a sequence of linear polynomials, $f_n(x) = m_n x + b_n$, by the recurrence $f_n(x) = f_{n-1}(f_{n-2}(x))$, $n \ge 3$, with initial conditions $f_1(x) = \frac{1}{2}x$ and $f_2(x) = \frac{1}{2}x + \frac{1}{2}$. Find a formula for m_n.

Extra credit: Find a formula for b_n.

Solution by Charles Ashbacher, Hiawatha, IA

We claim that $m_n = 1/2^{F_n}$ for $n \ge 1$. The proof is by induction on n. For the basis step, we have the given initial conditions, showing that the result is true for $n = 1$ and $n = 2$. Now assume

$$f_{k-1}(x) = \frac{1}{2^{F_{k-1}}}x + b_1 \quad \text{and} \quad f_k(x) = \frac{1}{2^{F_k}}x + b_2.$$

Then

$$\begin{aligned} f_{k+1}(x) &= \frac{1}{2^{F_k}}\left(\frac{1}{2^{F_{k-1}}}x + b_1\right) + b_2 = \frac{1}{2^{F_k}2^{F_{k-1}}}x + \frac{1}{2^{F_k}}b_1 + b_2 \\ &= \frac{1}{2^{F_k + F_{k-1}}}x + \frac{1}{2^{F_k}}b_1 + b_2 = \frac{1}{2^{F_{k+1}}}x + \frac{1}{2^{F_k}}b_1 + b_2 \end{aligned}$$

and the result follows for all n.

Bruckman receives extra credit for finding that

$$b_n = \sum_{k=1}^{F_{n-1}} \frac{1}{2^{\lfloor k\alpha \rfloor}}.$$

Strazdins reports that this sequence is studied in [1].

Reference

1. H. W. Gould, J. B. Kim, & V. E. Hoggatt, Jr. "Sequences Associated with t-ary Coding of Fibonacci's Rabbits." *The Fibonacci Quarterly* **15.4** (1977):311-18.

Also solved by Paul S. Bruckman, Charles K. Cook, L. A. G. Dresel, Russell Jay Hendel, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Late solutions to problems B-821 through B-824 were received from David Stone.

Errata: In the solution to problem B-830 (Feb. 1998, p. 89), in the second line of the proof of part (b), the subscript $19 \cdot 109$ should be $19 \cdot 108$ in three places. On the first line of page 90, "$n+1$" should read "$n+a$".

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