A NOTE ON A PAPER BY GLASER AND SCHÖFFL

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1. INTRODUCTION

Ducci sequences are sequences of integer n-tuples $\mathbf{u}_0, \mathbf{u}_1, \ldots$ generated by the relation $\mathbf{u}_{k+1} = D\mathbf{u}_k$, where $D(x_1, x_2, \ldots, x_n) = (|x_1 - x_2|, |x_2 - x_3|, \ldots, |x_n - x_1|)$. We say that $\mathbf{u}_0$ generates the Ducci sequence $\mathbf{u}_0, \mathbf{u}_1, \ldots$. It has been shown (e.g., in [3]) that every Ducci sequence reduces to a sequence of binary tuples $\mathbf{u}_k = (x_1, \ldots, x_n)$, where $x_i \in \{0, c\}$ for all i and some constant c. As $D(\lambda\mathbf{u}) = \lambda D\mathbf{u}$ for all $\lambda \geq 1$, it is customary to assume $c = 1$. At this point it is obvious that every Ducci sequence eventually forms a cycle, called a Ducci cycle.

Many aspects of Ducci sequences have been studied, such as the smallest k such that $\mathbf{u}_k$ is part of the Ducci cycle (this is also known as the "n-Number Game," see [4]). In this note we will only concern ourselves with the Ducci cycles themselves. We list some of the known results and conventions used in this note:

1. For binary tuples, D becomes the linear operator $D(x_1, x_2, \ldots, x_n) = (x_1 + x_2, x_2 + x_3, \ldots, x_n + x_1) \pmod 2$.
2. If $\mathbf{a}_0 = (0, 0, \ldots, 0, 1)$, then the Ducci sequence $\mathbf{a}_0, \mathbf{a}_1, \ldots$ is called the *basic Ducci sequence* and the resulting cycle the *basic Ducci cycle*. From now on, $\mathbf{a}_k$ will always denote a tuple in the basic Ducci sequence.
3. The *period* (also referred to as the *length*) of the basic Ducci cycle is denoted by $P(n)$ and many properties of $P(n)$ are given in [1] and [2]. When we speak of the period of a Ducci sequence, we actually mean the period of the Ducci cycle produced by that Ducci sequence.
4. If $n > 1$ is odd and there exists an M such that $2^M \equiv -1 \pmod n$, then n is said to be "with a -1"; otherwise, n is said to be "without a -1". This useful convention was introduced in [1] and used extensively in [2].

In this paper we generalize two of the results in [2]: first, we show how Pascal's triangle can be used to construct any tuple in the basic Ducci sequence; second, we determine, in general, the first tuple in the basic Ducci sequence that is part of the basic Ducci cycle. We also provide counter-examples of two other remarks in [2] concerning the number of Ducci cycles of maximum length and determine the cause for these errors (which do not, however, affect any other results in [2]).

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2. USING PASCAL'S TRIANGLE TO CONSTRUCT DUCCI SEQUENCES

Glaser and Schöffl described how Pascal's triangle can be used to find the first n-tuples of the basic Ducci sequence. If we assume the convention that

$$\binom{k}{r} = 0 \text{ if } r > k \text{ or } r < 0,$$

then this method can be expressed as the following theorem (Theorem 1 in [2]).

Theorem 1: The r^{th} entry of $\mathbf{a}_k$ is

$$x_r = \binom{k}{r+k-n} \pmod 2 \text{ if } k < n.$$

We shall prove a more general result.

Theorem 2: For all $k \geq 0$ the r^{th} entry of $\mathbf{a}_k$ is

$$x_r = \sum_{i \equiv r \pmod n} \binom{k}{i+k-n} \pmod 2.$$

Proof: From Theorem 1 we know that this is true for $k < n$. Assume it is also true for some $k = p$, so

$$x_r = \sum_{i \equiv r \pmod n} \binom{p}{i+p-n} \pmod 2.$$

Let us denote the r^{th} entry of $\mathbf{a}_p$ by x_r, and the r^{th} entry of $\mathbf{a}_{p+1}$ by x'_r. Now, if $k = p+1$, then we have two cases: If $r < n$, then we have

$$\begin{aligned} x'_r = x_r + x_{r+1} &= \sum_{i \equiv r} \binom{p}{i+p-n} + \sum_{i \equiv r+1} \binom{p}{i+p-n} \pmod 2 \\ &= \sum_{i \equiv r} \left[\binom{p}{i+p-n} + \binom{p}{i+1+p-n} \right] \pmod 2 \\ &= \sum_{i \equiv r} \binom{p+1}{i+p+1-n} \pmod 2; \end{aligned}$$

if $r = n$, then we have

$$\begin{aligned} x'_n = x_n + x_1 &= \sum_{i \equiv 0} \binom{p}{i+p-n} + \sum_{i \equiv 1} \binom{p}{i+p-n} \pmod 2 \\ &= \sum_{i \equiv 0} \left[\binom{p}{i+p-n} + \binom{p}{i+1+p-n} \right] \pmod 2 \\ &= \sum_{i \equiv 0} \binom{p+1}{i+p+1-n} \pmod 2. \end{aligned}$$

So, by induction, the theorem is true for all natural numbers k. □

This result suggests that we can construct the basic Ducci sequence by wrapping Pascal's triangle (mod 2) around a cylinder of circumference n and adding (mod 2) those entries that overlap. We demonstrate this below for the first seven tuples of the basic Ducci sequence for $n = 3$.

$$\begin{array}{c|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1
\end{array}
\rightarrow
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
+
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
+
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
0 & 0 & 1 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix},$$

giving

$$\begin{array}{ccc}
0 & 0 & 1 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{array}$$

as the first seven 3-tuples in the basic Ducci sequence.

3. TWO COUNTER-EXAMPLES

Just below their proof of Theorem 2 [2, p. 316], Glaser and Schöffl mention that for every n with a -1 there exists only one cycle of maximum length. This conclusion is false, as the counter-example (1, 0, 0, 1, 0, 0, 0, 0, 0) demonstrates; this tuple is not part of the basic Ducci sequence (which has maximum length) but is part of another Ducci sequence that also has maximum length. (This can be checked by computer: $n = 9$ is with a -1 and $P(9) = 63$. For values of $P(n)$ for $n \leq 165$, see [1].)

In Corollary 2 [2, p. 319], Glaser and Schöffl remark that for $n = 2^r - 1$ $(r \geq 2)$ there are exactly n different cycles of maximum length, namely, the n cyclic permutations of the basic Ducci cycle. Again, we offer a counter-example: (0, 1, 1, 1, 0, 0, 1) forms an 8th cycle of maximum length (shown below).

$$\begin{array}{ccccccc}
0 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}$$

The cycle given above is of maximum length as $P(7) = 7$.

Both of these errors arise from the misconception that the cycles of maximum length are precisely the basic cycles, an assumption that may have been implicit in earlier papers. The errors did not influence any other results in [2].

4. THE FIRST TUPLE OF A CYCLE

Glaser and Schöffl mentioned that Ehrlich [1] was able to describe the first n-tuple in the basic Ducci cycle if n is odd, but that nothing was known about the case in which n is even. They solved this problem for the cases $n = 2^r + 2^s$ and $n = 2^r - 2^s$, $r > s \geq 0$.

We give the general solution here, but first we must recall a result of Ludington Furno [3].

Let $n = 2^r k$, where k is odd.

Definition 1: We say the n-tuple $(x_1, \ldots, x_n)$ is r-even if

$$\sum_{i=0}^{k-1} x_{2^r i+j} \equiv 0 \pmod 2, \ \forall j = 1, \ldots, 2^r.$$

Let us count the number of n-tuples that are r-even. We can choose the first $n - 2^r$ entries arbitrarily, but the last 2^r entries are then uniquely determined by previous entries in order for the tuple to be r-even, so we have a total of 2^{n-2^r} r-even n-tuples. For example, if $n = 6 = 2^1 \cdot 3$, then the first four entries can be arbitrary, but the last two entries must have specific values in order for the tuple to be 1-even. So we have a total of $2^4 = 16$ 1-even 6-tuples.

Theorem 3 (Ludington Furno): An n-tuple is in a cycle if and only if it is r-even.

From this, follows

Theorem 4: $\mathbf{a}_{2^r} = (0, \ldots, 0, 1, \underbrace{0, \ldots, 0}_{2^r-1 \text{ zeros}}, 1)$ is the first n-tuple in the basic Ducci cycle.

Proof: Obviously, $\mathbf{a}_{2^r}$ is r-even, so it is in a cycle. As it is the $2^{r\text{th}}$ row in Pascal's triangle, it is in the basic Ducci cycle. To show that $\mathbf{a}_t = (x_1, \ldots, x_n)$ is not in a cycle for $t < 2^r$, we need only note that $x_1 = x_2 = \cdots = x_{n-2^r} = 0$; thus,

$$S = \sum_{i=0}^{k-1} x_{2^r i+j} \leq 1 \text{ for all } j = 1, \ldots, 2^r.$$

But $S \neq 0$ for at least one j; therefore, $\mathbf{a}_t$ is not r-even (therefore, not in a cycle) for $t < 2^r$. □

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