ON THE DEGREE OF THE CHARACTERISTIC POLYNOMIAL OF POWERS OF SEQUENCES

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In [2], Cooper and Kennedy considered the following question: If $\{U_n\}$ is a sequence satisfying a third-order linear recurrence, what is the degree of the recurrence satisfied by the sequence $\{(U_n)^2\}$? They gave the answer as 6. They then asked if there is a similar result for the sequence $\{(U_n)^3\}$, tossing this question out as a research problem.

In [4], Prodinger answered this latter question in the affirmative, along with the more general question dealing with linear recurrences of any order and arbitrary powers of the original sequence. In the case of the familiar Fibonacci (or Lucas) sequence (where the original sequence satisfies a second-order linear recurrence), Prodinger displayed the recurrences satisfied by $\{(F_n)^k\}$ (or $\{(L_n)^k\}$) for $k = 1, 2, 3, 4, 5, 6$, showing that such recurrences are all linear and of order $(k+1)$. As Cooper and Kennedy had observed in [2], these latter recurrences had been obtained by D. Jarden [3] and are special cases of the following formula:

$$\sum_{j=0}^{k+1}(-1)^{j(j+1)/2}[k+1, j]_F(F_{n-j})^k = 0, \quad k = 1, 2, \ldots; n \text{ any integer.} \tag{1}$$

In this formula, the quantities $[k, j]_F$ are the *Fibonomial coefficients* defined by:

$$[k, j]_F \equiv [k!]_F / \{[j!]_F[(k-j)!]_F\},$$

where $0 \le j \le k$, with $[m!]_F \equiv F_1F_2F_3\ldots F_m$, $m \ge 1$, and $[0!]_F = 1$. A table of Fibonomial coefficients is provided in Brother Alfred Brousseau's compendium [1]. The formula in (1) is a special case of a more general formula (omitted here) due to Jarden and given in [3], involving certain sequences satisfying a second-order linear recurrence.

It should be added that although Prodinger demonstrated the existence of the order of certain linear recurrences in more general cases than was explored by Cooper and Kennedy, he did not actually derive an exact expression for such order. We rectify this omission in this paper, and extend such result to an even more general situation.

It seems natural to ask whether we can find similar results for the most general type of sequence satisfying a linear recurrence. It will be noted from recurrence theory that any sequence satisfying a linear recurrence possesses a characteristic polynomial of a certain degree with eigenvalues (also known as characteristic roots) of possibly multiple order. In general, such sequence is *nonlinear*. More specifically, we consider a sequence $\{U_n\}$ of the following known form:

$$U_n = \sum_{j=1}^{m}\theta_j(n)(\alpha_j)^n, \tag{2}$$

where the $\theta_j(n)$ are given polynomials in n of degree r_j (with $r_j \ge 0$), and the α_j's are distinct given constants. Such sequences are denoted as *polynomial sequences*. Incidentally, we note that, from the known expression for U_n, we may immediately write the characteristic polynomial $P_1(z)$ of the sequence, namely:

$$P_1(z) = \prod_{j=1}^{m} (z-\alpha_j)^{1+r_j}. \tag{3}$$

Observe that the sequence $\{(U_n)^k\}$ $(k = 1, 2, 3, \ldots)$ also possesses a characteristic polynomial, which we denote by $P_k(z)$. We let R_k represent the degree of $P_k(z)$. By definition of the characteristic polynomial, $P_k(z)$ is the *minimum* polynomial such that $P_k(E)(U_n^k) = 0$ (where E is the unit shift operator, i.e., $Ex_n = x_{n+1}$). In other words, R_k is the *order* of the recurrence satisfied by the k^{th} power of the original sequence. Our task is thus to determine k^{th} for $k = 1, 2, 3, \ldots$.

Indeed (given (3)), we immediately determine that

$$R_1 = \sum_{j=1}^{m} (1+r_j). \tag{4}$$

We claim the following main result:

Theorem:

$$R_k = (R_1 - m)\binom{k+m-1}{k-1} + \binom{k+m-1}{k}. \tag{5}$$

In particular, if $r_j = 0$ for $j = 1, 2, \ldots, m$, then the characteristic roots are of order one and $R_1 = m$; in this case,

$$R_k = \binom{k+m-1}{k}. \tag{6}$$

This latter result is clearly a corollary of the Theorem. If the original recurrence has characteristic roots of single order, then the characteristic roots of the "power recurrence" are also of single order. For the particular case where $R_1 = m = 2$. we obtain Prodinger's implied result: $R_k = k+1$.

Proof of (5): We begin by expanding the k^{th} power of the given expression for U_n, using the multinomial theorem:

$$(U_n)^k = \sum_{S(m,k)} \binom{k}{i_1, i_2, \ldots, i_m} \{\theta_1(n)(\alpha_1)^n\}^{i_1} \{\theta_2(n)(\alpha_2)^n\}^{i_2} \cdots \{\theta_m(n)(\alpha_m)^n\}^{i_m},$$

where $S(m, k) = \{(i_1, i_2, \ldots, i_m) : i_1 + i_2 + \cdots + i_m = k,\ 0 \le i_j \le k,\ j = 1, 2, \ldots, m\}$, and $\binom{k}{i_1, i_2, \ldots, i_m}$ is the multinomial coefficient evaluated as $k!/\{(i_1)!(i_2)!\ldots(i_m)!\}$. Note that

$$\text{degree}[\{\theta_1(n)\}^{i_1}\{\theta_2(n)\}^{i_2} \cdots \{\theta_m(n)\}^{i_m}] = \sum_{j=1}^{m} r_j i_j.$$

We see that $P_k(z) = \prod_{S(m,k)} \{z - (\alpha_1)^{i_1}(\alpha_2)^{i_2} \cdots (\alpha_m)^{i_m}\}^{E(i_1, i_2, \ldots, i_m)}$, where

$$E(i_1, i_2, \ldots, i_m) = 1 + \sum_{j=1}^{m} r_j i_j. \tag{7}$$

Therefore,

$$R_k = \sum_{S(m,k)} E(i_1, i_2, \ldots, i_m). \tag{8}$$

It remains to evaluate the last expression. Towards this end, we employ a pair of lemmas. For convenience, we let $U(m, k)$ denote

$$|S(m,k)| = \sum_{S(m,k)} 1,$$

the cardinality of $S(m, k)$, and

$$V(m,k) = \sum_{S(m,k)} i_1.$$

It follows (by symmetry) that

$$V(m,k) = \sum_{S(m,k)} i_j, \quad j = 1, 2, \ldots, m.$$

Therefore, we see from (7) and (8) that $R_k = U(m,k) + V(m,k)\sum_{j=1}^{m} r_j$, or

$$R_k = U(m,k) + (R_1 - m)V(m,k). \tag{9}$$

Lemma 1:

$$U(m,k) = \binom{k+m-1}{k}. \tag{10}$$

Proof (by induction on m): Let K denote the set of $m \geq 1$ such that (10) is true (k being treated as fixed). Since $S(1,k) = \{k\}$, we see that $U(1,k) = 1 = \binom{k}{k}$; therefore, $1 \in K$. Suppose $1, 2, \ldots, m \in K$. Now $S(m+1, k)$ consists of those vectors in ε^{m+1} which have their first component equal to i_1 and the remaining vector (an element of ε^m) equal to a vector in $S(m, k-i_1)$. Since i_1 varies from 0 to k, inclusive, it follows that

$$U(m+1,k) = \sum_{j=0}^{k} U(m,j). \tag{11}$$

By the inductive hypothesis,

$$U(m+1,k) = \sum_{j=0}^{k}\binom{j+m-1}{j} = \sum_{j=m-1}^{k+m-1}\binom{j}{m-1} = \binom{k+m}{m} = \binom{k+m}{k}.$$

We see that this result is the statement of (10) for $(m+1)$. Thus,

$$1, 2, \ldots, m \in K \Rightarrow 1, 2, \ldots, m, m+1 \in K.$$

Induction completes the proof. □

Lemma 2:

$$V(m,k) = \binom{k+m-1}{k-1}. \tag{12}$$

Proof: Reasoning as in the proof of Lemma 1,

$$V(m,k) = \sum_{j=0}^{k}(k-j)U(m-1,j).$$

Using the result of Lemma 1 and standard combinatorial manipulations,

$$V(m,k) = \sum_{j=0}^{k-1}(k-j)\binom{j+m-2}{j} = k\sum_{j=m-2}^{k+m-3}\binom{j}{m-2} - (m-1)\sum_{j=m-1}^{k+m-3}\binom{j}{m-1}.$$

Then

$$V(m,k)=k\binom{k+m-2}{m-1}-(m-1)\binom{k+m-2}{m}=\binom{k+m-1}{k-1},$$

after simplification. Substituting the results of Lemmas 1 and 2 into (9) yields the Theorem. □

As an illustration of our formula, consider the original sequence to be $\{U_n\}=\{n^2\}$. In this case, $P_1(z)=(z-1)^3$; hence, $m=1$, $\alpha_1=1$, $r_1=2$, $R_1=3$. In other words, U_n satisfies the third-order linear recurrence: $U_{n+3}-3U_{n+2}+3U_{n+1}-U_n=0$. Then $(U_n)^k=n^{2k}$, for which the characteristic polynomial $P_k(z)=(z-1)^{2k+1}$, and $R_k=2k+1$. In particular, $R_2=5\neq 6$. Thus, the result of Cooper and Kennedy [2] needs to be modified somewhat. Although it is true that the square of a sequence satisfying a third-order linear recurrence satisfies a linear recurrence of order 6, it may happen that such square sequence in fact satisfies a linear recurrence of order 5; in such case, its characteristic (i.e., *minimal*) polynomial has degree 5, rather than 6. Similar anomalies occur when the original recurrence has characteristic roots or multiplicity greater than one. The main theorem given in this paper treats all such cases with full generality, giving the *minimum* order of the appropriate recurrence. It needs to be emphasized, however, that this order is known only if the characteristic roots of the original sequence and their multiplicities are known in advance (or, equivalently, if the characteristic polynomial is known in advance, along with all of its factors).

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