A NEW RECURRENCE FORMULA FOR BERNOULLI NUMBERS

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1. INTRODUCTION

Let B_n be the Bernoulli numbers defined by the expansion

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}.$$

Kaneko [3] proved a recurrence formula for B_n,

$$\sum_{j=0}^{n}\binom{n+1}{j}(n+j+1)B_{n+j}=0 \quad (n\geq 1), \tag{1}$$

which needs only half the number of terms to calculate B_{2n} in comparison with the usual recurrence (cf. [5], §15, Lemma 1):

$$\sum_{j=0}^{n}\binom{n+1}{j}B_j=0 \quad (n\geq 1). \tag{2}$$

The aim of this paper is to prove the following recurrence formula that yields Kaneko's formula when $m=n$ and also the usual one.

Theorem: For nonnegative integers m and n with $m+n>0$, we have the formula

$$\sum_{j=0}^{m}\binom{m+1}{j}(n+j+1)B_{n+j}+(-1)^{m+n}\sum_{k=0}^{n}\binom{n+1}{k}(m+k+1)B_{m+k}=0.$$

As an application of our theorem, we can derive the Kummer congruence. The proof of our theorem uses the Volkenborn integral (whose properties are found in [4]).

2. PROOF OF THE THEOREM

Let p be a prime number and let $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the ring of p-adic integers and the field of p-adic numbers, respectively. For any uniformly differentiable function $f:\mathbb{Z}_p\to\mathbb{Q}_p$, we define the Volkenborn integral of f by

$$\int_{\mathbb{Z}_p}f(x)\,dx:=\lim_{n\to\infty}p^{-n}\sum_{j=0}^{p^n-1}f(j).$$

In particular, the Bernoulli number B_n is given by the formula

$$B_n=\int_{\mathbb{Z}_p}x^n\,dx. \tag{3}$$

Let m and n be nonnegative integers with $m+n>0$. If we define the polynomial function $G(x)$ on $\mathbb{Z}_p$ by

$$G(x):=x^{m+1}(x-1)^{n+1}+(-1)^{n+m}x^{n+1}(x-1)^{m+1},$$

then we have $G'(x+1)=-G'(-x)$. Therefore, we have

$$\int_{\mathbb{Z}_p} G'(x+1)\,dx=0$$

(see [4], Proposition 55.7). To calculate the left-hand side of this equation, we write $G(x+1)$ in the form

$$G(x+1)=\sum_{j=0}^{m+1}\binom{m+1}{j}x^{n+j+1}+(-1)^{m+n}\sum_{k=0}^{n+1}\binom{n+1}{k}x^{m+k+1}.$$

Applying formula (3) to the derivative $G'(x+1)$, we obtain

$$\sum_{j=0}^{m+1}\binom{m+1}{j}(n+j+1)\,B_{n+j}+(-1)^{m+n}\sum_{k=0}^{n+1}\binom{n+1}{k}(m+k+1)\,B_{m+k}=0.$$

Since $B_j=0$ for odd $j>1$ and, hence, the terms involving B_{n+m+1} vanish, this gives the formula of our theorem.

Remark: For a positive integer s, we consider the polynomial function

$$F(x):=x^{m+1}(x-s-1)^{n+1}+(-1)^{n+m}x^{n+1}(x-s-1)^{m+1}$$

on $\mathbb{Z}_p$. Then we have $F(x+s+1)=F(-x)$. It follows from Propositions 55.5 and 55.7 in [4] that

$$\begin{aligned}\int_{\mathbb{Z}_p} F'(x+s+1)\,dx &= \sum_{j=1}^{s}\int_{\mathbb{Z}_p}(F'(x+j+1)-F'(x+j))\,dx+\int_{\mathbb{Z}_p}F'(x+1)\,dx\\ &=\sum_{j=1}^{s}F''(j)+\int_{\mathbb{Z}_p}F'(-x)\,dx\\ &=\sum_{j=1}^{s}F''(j)-\int_{\mathbb{Z}_p}F'(x+s+1)\,dx.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\sum_{j=0}^{n}\binom{n+1}{j}(s+1)^{n+1-j}(m+j+1)\,B_{m+j}\\ &\qquad+(-1)^{m+n}\sum_{k=0}^{m}\binom{m+1}{k}(s+1)^{m+1-k}(n+k+1)\,B_{n+k}=\frac{1}{2}\sum_{j=1}^{s}F''(j).\end{aligned}$$

If $s=1$ and $m=n$, then we have the formula

$$\sum_{k=0}^{n}\binom{n+1}{k}2^{n+1-k}(n+k+1)\,B_{n+k}=(-1)^n(n+1)\quad(n\geq 1),$$

which resembles the well-known formula (see [2], §15, Theorem 1)

$$\sum_{k=0}^{n}\binom{n+1}{k}2^{n+1-k}B_k=n+1.$$

3. SEVERAL CONSEQUENCES

We shall derive the usual formula (2) from our theorem. If $m=0$, we obtain

$$(-1)^n \sum_{k=0}^{n} \binom{n+1}{k}(k+1)B_k = -(n+1)B_n \quad (n \ge 1). \tag{4}$$

For convenience, we put

$$C_n := (-1)^n (n+2) \sum_{k=0}^{n} \binom{n+1}{k} B_k .$$

It is obvious that the usual formula is equivalent to $C_n = 0$ for $n \ge 1$. Substituting the identity

$$(k+1)\binom{n+1}{k} = (n+2)\binom{n+1}{k} - (n+1)\binom{n}{k}$$

into equation (4) yields

$$C_n + C_{n-1} = -(n+1)(1-(-1)^n)B_n.$$

Since $B_j = 0$ for odd $n > 1$, the right-hand side of this equation vanishes for $n \ge 2$. It is clear that $C_1 = 0$, hence $C_n = 0$ for $n \ge 1$.

We next show Kummer's congruence

$$\frac{B_n}{n} \equiv \frac{B_{n+(p-1)}}{n+(p-1)} \pmod{p\mathbb{Z}_p}$$

when p is a prime number with $p \ge 5$ and n is an integer with $1 \le n \le p-2$. Our argument is similar to Agoh's argument [1].

If $m = p-1$, the formula of our theorem is

$$\sum_{j=0}^{p-1} \binom{p}{j}(n+j+1)B_{n+j} + (-1)^n \sum_{k=0}^{n} \binom{n+1}{k}(p+k)B_{p-1+k} = 0. \tag{5}$$

Note that $1 \le n+j < 2(p-1)$ for $0 \le j \le p-1$. From the well-known fact (see von Staudt and Clausen [2], §15, Corollary to Theorem 3) that

$$pB_{n+j} \equiv \begin{cases} -1 \pmod{p\mathbb{Z}_p} & \text{if } n+j = p-1, \\ 0 \pmod{p\mathbb{Z}_p} & \text{otherwise,} \end{cases} \tag{6}$$

we have

$$\binom{p}{j}(n+j+1)B_{n+j} \equiv 0 \pmod{p\mathbb{Z}_p}$$

for $j \ne 0$. Thus, equation (5) yields

$$(n+1)B_n + (-1)^n \sum_{k=0}^{n} \binom{n+1}{k}(p+k)B_{p-1+k} \equiv 0 \pmod{p\mathbb{Z}_p}.$$

Applying congruence (6) to the above, we have

$$(n+1)B_n + (-1)^n \sum_{k=1}^{n} \binom{n+1}{k} kB_{p-1+k} \equiv (-1)^n \pmod{p\mathbb{Z}_p}.$$

We remark that combining equation (2) with (4) gives

$$(n+1)B_n = (-1)^{n+1}\sum_{k=1}^{n}\binom{n+1}{k}(k+p-1)B_k + (-1)^{n+1}(p-1)B_0.$$

Since $B_0 = 1$, we have

$$\sum_{k=1}^{n}\binom{n+1}{k}(kB_{p-1+k} - (k+p-1)B_k) \equiv 0 \pmod{p\mathbb{Z}_p}.$$

From these congruences, we have by induction on n that

$$nB_{p-1+n} \equiv (p+n-1)B_n \pmod{p\mathbb{Z}_p}$$

for $1 \le n \le p-2$. This yields Kummer's congruence as desired.

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