ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A.P. Hillman
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within two months of publication.

B-82 *Proposed by Nanci Smith, University of New Mexico, Albuquerque, N.M.*

Describe a function $g(n)$ having the table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	...
g(n)	0	1	1	2	1	2	2	3	1	2	2	3	2	...

B-83 *Proposed by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada*

Show that $F_n^2 + F_{n+4}^2 = F_{n+1}^2 + F_{n+3}^2 + 4F_{n+2}^2$.

B-84 *Proposed by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada*

The Fibonacci polynomials are defined by $f_1(x) = 1$, $f_2(x) = x$,

$$f_{n+1}(x) = xf_n(x) + f_{n-1}(x), \qquad n > 1 .$$

If $z_r = f_r(x) + f_r(y)$, show that z_r satisfies

$$z_{n+4} - (x+y)z_{n+3} + (xy-2)z_{n+2} + (x+y)z_{n+1} + z_n = 0 .$$

B-85 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Find compact expressions for:

(a) $$F_2^2 + F_4^2 + F_6^2 + \ldots + F_{2n}^2$$

(b) $$F_1^2 + F_3^2 + F_5^2 + \ldots + F_{2n-1}^2 .$$

B-86 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that the squares of every third Fibonacci number satisfy

$$y_{n+3} - 17y_{n+2} + 17y_{n+1} - y_n = 0 \quad .$$

B-87 *Proposed by A.P. Hillman, University of New Mexico, Albuquerque, N.M.*

Prove the identity in

$$\sum_{k=0}^{n} \left[\frac{(-1)^{n-k}}{k!\,(n-k)!} \prod_{j=0}^{n} (x_j + k)\right] = \binom{n+1}{2} + \sum_{j=0}^{n} x_j \quad .$$

SOLUTIONS

AN N-TUPLE INTEGRAL

B-70 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Denote x^a by ex(a). Show that the following expression, containing n integrals,

$$\int_0^1 \mathrm{ex}\left(\int_0^1 \mathrm{ex}\left(\int_0^1 \mathrm{ex}\left(\ldots \int_0^1 \mathrm{ex}\left(\int_0^1 x\,dx\right)dx\right)\ldots dx\right)dx\right)dx$$

equals F_{n+1}/F_{n+2}, where F_n is the n-th Fibonacci number.

Solution by John Wessner, Melbourne, Florida

Let I_n denote the n-th such integral. Then

$$I_1 = \int_0^1 x\,dx = 1/2 \quad .$$

Let us assume that $I_{n-1} = F_n/F_{n+1}$, in which case

$$I_n = \int_0^1 x^{F_n/F_{n+1}}\,dx = \left\{(F_n/F_{n+1}) + 1\right\}^{-1}$$

$$= \left\{(F_n + F_{n+1})/F_{n+1}\right\}^{-1} = F_{n+2}/F_{n+1} \quad ,$$

which was to be shown.

Also solved by R.J. Hursey, Jr; M.N.S. Swamy; Howard L. Walton; David Zeitlin; and the proposer

B-71 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Find $a^{-2} + a^{-3} + a^{-4} + \ldots,$ where $a = (1 - 5)/2$.

Solution by John W. Milsom, Slippery Rock State College, Slippery Rock, Penna.

If $S = a^{-2} + a^{-3} + a^{-4} + \ldots,$ then $a^2 S = 1 + a^{-1} + a^{-2} + \ldots$.

Subtracting the first equation from the second,

$$a^2 S - S = 1 + a^{-1}$$

$$S = (1 + a^{-1})/(a^2 - 1)$$

$$S = 1/[a(a - 1)] \; .$$

Using $a = (1 + \sqrt{5})/2$, we find that $S = 1$. If you use $a = (1 + 5)/2 = = 6/2 = 3$, as the problem reads, the result is $S = 1/6$.

Also solved by R.J. Hursey, Jr; Sidney Kravitz; M.N.S. Swamy; C.W. Trigg; Howard L. Walton; John Wessner; David Zeitlin; and the proposer.

ADDING RABBITS?

B-72 *Proposed by J.A.H. Hunter, Toronto, Canada*

Each distinct letter in this simple alphametic stands for a particular and different digit. We all know how rabbits link up with the Fibonacci series, so now evaluate our RABBITS.

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RABBITS
   BEAR
RABBITS
     AS
A SERIES
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Solution by Charles W. Trigg, San Diego, California

By the first column from the left, $0 < R < 5$. By the seventh column, $2S + R = 10k$, so $S \neq 0$, and R is even. That is, $R = 2$ or 4.

By the fourth column, $3B + 1 = R$, so B is odd.

With these and the obvious relations from the other columns we can proceed to establish the values of the letters in the order given in the table below:

R	B	A	S	E	T	I
2	7	4	9	6	3	3
					8	2
4	1	9	8	2	6	5

Since the first two sets contain duplicate digits, the third set is the unique solution. Thus

$$\begin{array}{r} 4911568 \\ 1294 \\ 4911568 \\ 98 \\ \hline 9824528 \end{array}$$

That is, RABBITS = 4911568, which just goes to show what 2 rabbits can do.

Also solved by Murray Berg; Rudolph W. Castown; Sidney Kravitz; John W. Milsom; Azriel Rosenfeld; and the proposer.

DOUBLE SUMS

B-73 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Prove that

$$\sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{k}\binom{k+r-j-1}{j} = 1 + \sum_{m=0}^{2n+r-2}\sum_{p=0}^{m}\binom{m-p-1}{p},$$

where $\binom{n}{r} = 0$ for $n < r$.

Solution by David Zeitlin, Minneapolis, Minnesota

The given identity is valid only for $r \le n + 1$. Since

$$F_{n-1} = \sum_{j=0}^{[n/2]}\binom{n-j}{j},\quad \sum_{k=0}^{n} F_k = F_{n+2} - 1,\text{ and } \sum_{k=0}^{n}\binom{n}{k}F_{k+m} = F_{2n+m},$$

we have

$$1 + \sum_{m=0}^{2n+r-2}\sum_{p=0}^{m}\binom{m-p-1}{p} = 1 + \sum_{m=0}^{2n+r-2} F_m = F_{2n+r};$$

while for $r \le n + 1$, we have

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{n} \binom{k+r-j-1}{j} = \sum_{k=0}^{n} \binom{n}{k} F_{k+r} = F_{2n+r} .$$

Also solved by the proposer.

FIBONACCI POLYNOMIALS

B-74 *Proposed by M.N.S. Swamy, University of Saskatchewan, Regina, Canada*

The Fibonacci polynomial $f_n(x)$ is defined by $f_1 = 1$, $f_2 = x$, and $f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$ for $n > 2$. Show the following:

(a) $$x \sum_{r=1}^{n} f_r(x) = f_{n+1} + f_n - 1 .$$

(b) $$f_{m+n+1} = f_{m+1} f_{n+1} + f_m f_n .$$

(c) $$f_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1} ,$$

where $[k]$ is the greatest integer not exceeding k. Hence show that the n-th Fibonacci number

$$F_n = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} .$$

Solution by David Zeitlin, Minneapolis, Minnesota

(a) Assuming the relation to be true for $n = n$, we have

$$x \sum_{r=1}^{n+1} f_r(x) = xf_{n+1} + (f_{n+1} + f_n - 1)$$
$$= f_{n+2} + f_{n+1} - 1 ,$$

and the result now follows by mathematical induction.

(b) Using formula (6) in my paper, "On summation formulas for Fibonacci and Lucas numbers," this Quarterly, vol. 2, 1964, No. 2, p. 105, we have (since $f_0 = 0$)

$$\frac{f_{m+1} + f_m \cdot y}{1 - xy - y^2} = \sum_{n=0}^{\infty} f_{m+n+1}\, y^n , \tag{1}$$

$$\frac{f_{m+1}}{1 - xy - y^2} = \sum_{n=0}^{\infty} f_{m+1}\, f_{n+1}\, y^n , \tag{2}$$

$$\frac{f_m \cdot y}{1 - xy - y^2} = \sum_{n=0}^{\infty} f_m f_n\, y^n . \tag{3}$$

Since (1) = (2) + (3), the result follows by equating coefficients of y^n.

(c) We note that

$$\frac{y}{1-xy-y^2} = \sum_{n=0}^{\infty} f_n(x) y^n$$

and recall that

$$\frac{1}{1-2tz+z^2} = \sum_{n=0}^{\infty} U_n(t) z^n ,$$

where $U_n(t)$ is the Chebyshev polynomial of the second kind defined by

$$U_n(t) = \sum_{j=0}^{[n/2]} (-1)^j \binom{n-j}{j} (2t)^{n-2j} . \tag{4}$$

with $i^2 = -1$, we see that for $z = iy$ and $t = x/2i$, we have

$$\frac{1}{1-xy-y^2} = \sum_{n=0}^{\infty} i^n\, U_n\left(\frac{x}{2i}\right) y^n ,$$

and thus $f_{n+1}(x) = i^n\, U_n(x/2i)$, the desired result, using (4).

Since $F_n = f_n(1)$, we obtain

$$F_n = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} .$$

Also solved by the proposer.

DERIVATIVES OF FIBONACCI POLYNOMIALS

B-75 *Proposed by M.N.S. Swamy, University of Saskatchewan, Regina, Canada*

Let $f_n(x)$ be as defined in B-74. Show that the derivative

$$f'_n(x) = \sum_{r=1}^{n-1} f_r(x)\, f_{n-r}(x) \quad \text{for } n > 1.$$

Solution by David Zeitlin, Minneapolis, Minnesota

If we differentiate with respect to x the identity

$$\frac{y}{1-xy-y^2} = \sum_{n=0}^{\infty} f_n(x)y^n ,$$

we obtain

$$\sum_{n=0}^{\infty} f'_n(x)y^n = \left(\frac{y}{1-xy-y^2}\right)^2 = \left(\sum_{n=0}^{\infty} f_n(x)y^n\right)^2$$

$$= \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} f_r(x)f_{n-r}(x)\right] y^n .$$

If we equate coefficients of y^n, we obtain

$$f'_n(x) = \sum_{r=0}^{n} f_r(x)f_{n-r}(x)$$

$$= \sum_{r=1}^{n-1} f_r(x)f_{n-r}(x) \qquad (\text{since } f_0(x) = 0) .$$

Also solved by Lawrence D. Gould and the proposer.

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