THE $3x+1$ PROBLEM AND DIRECTED GRAPHS

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0. INTRODUCTION

Let $\mathbf{Z}$ denote the set of integers, $\mathbf{P}$ denote the positive integers, and $\mathbf{N}$ denote the nonnegative integers. Define the *Collatz mapping* $T: 2\mathbf{N}+1 \to 2\mathbf{N}+1$ by $T(x) = (3x+1)/2^j$, where $2^j \mid 3x+1$ but $2^{j+1} \nmid 3x+1$. The famous $3x+1$ *Conjecture*, or *Collatz Problem*, asserts that, for any $x \in 2\mathbf{N}+1$, there exists $k \in \mathbf{N}$ satisfying $T^k(x) = 1$, where T^k denotes k compositions of the function T. This paper's version of the Collatz mapping is also found in [4], whereas the most commonly used version is given in the comprehensive survey of Lagarias [6] and the research monograph of Wirsching [9]. It is obvious that our formulation of the $3x+1$ Conjecture is equivalent to those given in [6] and [9].

It is natural to study the $3x+1$ Conjecture in terms of the directed graph $G_{2\mathbf{N}+1}$ with vertices $2\mathbf{N}+1$ and directed edges from x to $T(x)$. A portion of this graph, known as the *Collatz graph* [6], is displayed in Figure 1. A slightly different version of the Collatz graph, which includes the positive even integers, is presented in [6], whereas $G_{2\mathbf{N}+1}$ excludes these with the purpose of making upcoming properties of certain vertices more transparent.

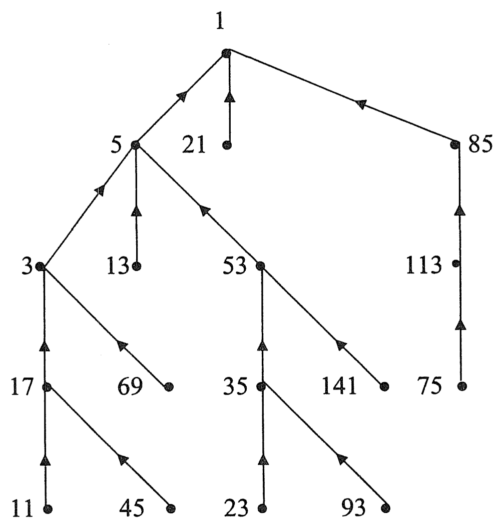

Figure 1. The Collatz Graph $G_{2\mathbf{N}+1}$ ($T^4(x) = 1, x < 150$)

A directed graph is said to be *weakly connected* if it is connected when viewed as an undirected graph, and we will call a pair of vertices weakly connected if they are connected by an undirected path. Using these graph-theoretical considerations, the $3x+1$ Conjecture can be restated as follows:

$3x+1$ Conjecture (1st form): The Collatz graph is weakly connected.

Our immediate goal is to identify a collection of vertices of $G_{2\mathbf{N}+1}$ which have a certain connectivity property (Section 1). We then use this result to analyze new directed graphs with vertex sets contained in $2\mathbf{N}+1$ for which weak connectivity also implies truth of the $3x+1$ Conjecture (Sections 2 and 3). Some conditions under which vertices of these new graphs are weakly connected are given. Certain numbers x satisfying the condition that $T^2(x)=1$ are discussed in Section 4. (A different characterization of some positive integers satisfying $T^k(x)=1$ can be found in [2].) In Section 4, we also prove the facts that cycles and divergent trajectories in our new graphs induce cycles and divergent trajectories in the original Collatz graph.

1. VERTICES WITH A SPECIAL CONNECTIVITY PROPERTY

To identify our vertex set, we need a few preliminaries. For $x \in 2\mathbf{N}+1$, the *total stopping time* of x, denoted $\sigma(x)$, is the least whole number k satisfying $T^k(x)=1$. (If no such k exists, set $\sigma(x)=\infty$.) Define the binary relation $\approx$ on $2\mathbf{N}+1$ as follows: $x \approx y$ if and only if there exists $k \in \mathbf{N}$ with $k \le \min(\sigma(x), \sigma(y))$ satisfying $T^k(x)=T^k(y)$. Clearly, $\approx$ is an equivalence relation, hence each $x \in 2\mathbf{N}+1$ belongs to an equivalence class C_x. Observe that $\sigma(x)=\sigma(y)<\infty$ implies that $x \approx y$, and furthermore, the set $L_k=\{x \in 2\mathbf{N}+1 \mid \sigma(x)=k\}$ is an equivalence class under $\approx$.

Progress has been made recently in determining the density of positive integers x satisfying $\sigma(x)<\infty$. The strongest known result is in [3], where it is shown that, if $\pi(x)$ counts the number of integers n satisfying $|n|<x$ and $\sigma(n)<\infty$, then, for all sufficiently large x, $\pi(x) \ge x^{.81}$. Important groundwork for this result was provided by Krasikov [5], who used a scheme of difference inequalities to show that $\pi(x) \ge x^{3/7}$. A stochastic approach for analyzing total stopping times is presented in [7], and a thorough summary of currently known total stopping time results can be found in [9].

It also bears mentioning that, throughout the literature, there is a distinct difference between *stopping time* and *total stopping time*. The stopping time of x is defined to be the least positive integer k for which $T^k(x)<x$. The most important stopping time result is given in [8], where it is shown that the density of positive integers with finite stopping time is 1.

We are not ready to state and prove our first result, which can also be found in [1]. The proof reveals properties of certain vertices of the Collatz graph which are useful later; therefore, it is presented here.

Theorem 1: If $x>5$ is the smallest element in C_x, then there exists $n \in \mathbf{P}$ such that $T^n(x)=T^n(2x+1)$.

Proof: Let A_n denote the arithmetic progression $\{2^{n+2}m+2^n-1\}_{m=0}^{\infty}$, and let B_n denote the arithmetic progression $\{2^{n+2}m+2^{n+1}+2^n-1\}_{m=0}^{\infty}$. If we let $S_1=\bigcup_{n\in 2\mathbf{N}+1}(A_n)$, $S_2=\bigcup_{n\in 2\mathbf{P}}(B_n)$, $S_3=\bigcup_{n\in 2\mathbf{P}}(A_n)$, and $S_4=\bigcup_{n\in 2\mathbf{N}+1}(B_n)$, it is easy to verify that $\{S_1, S_2, S_3, S_4\}$ is a partition of $2\mathbf{N}+1$. We now show that $x \in S_3 \cup S_4$ is impossible. If $x \in S_3$, write $x=2^{n+2}m+2^n-1$, where n is even, and if $x \in S_4$ and $n=1$, choose y satisfying $4y+1=x$, else choose y satisfying $2y+1=x$. In all cases, a straightforward computation, taking parity of n into consideration when necessary, shows that $T^n(x)=T^n(y)$. Hence $y \approx x$ with $y<x$, contradicting the fact that x is smallest in its equivalence class. Therefore, $x \notin S_3 \cup S_4$, so $x \in S_1 \cup S_2$. If $x \in S_1$, write $x=2^{n+2}m+2^n-1$ with n odd, and if $x \in S_2$, write $x=2^{n+2}m+2^{n+1}+2^n-1$ with n even. Again applying the Collatz function n times and taking parity of n into account, we obtain $T^n(x)=T^n(2x+1)$. $\square$

Corollary 1 follows easily from Theorem 1.

Corollary 1: If x is the smallest element in L_k, then the vertices x and $2x+1$ of $G_{2\mathbf{N}+1}$ are weakly connected.

2. REDUCING THE VERTEX SET OF THE COLLATZ GRAPH

We now construct a new directed graph whose vertices are the smallest elements of the equivalence classes under $\approx$. The primary tool used is a mapping $\hat{T}$ induced by the Collatz mapping. The construction has the advantage of reducing the set of vertices of the Collatz graph, but the disadvantage of sacrificing some information about $T(x)$.

Let $M = \{x \in 2\mathbf{N}+1 | x \le y \text{ for all } y \in C_x\}$. For $S \subseteq \mathbf{P}$, define $\chi(S)$ to be the smallest element of S. Define $\hat{T}: M \to M$ by $\hat{T}(m) = \chi(C_{T(m)})$. Due to the fact that every vertex of the Collatz graph is weakly connected to some $m \in M$, the following statement is equivalent to the $3x+1$ Conjecture.

$3x+1$ Conjecture (2^{nd} form): The directed graph G_M with vertices M and directed edges from m to $\hat{T}(m)$ is weakly connected.

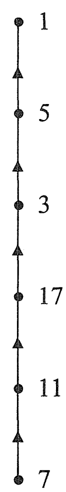

FIGURE 2. The Graph G_M $(\sigma(x) \le 5)$

A portion of G_M is displayed in Figure 2. The graph G_M, in effect, collapses the vertices of $G_{2\mathbf{N}+1}$ whose trajectories enter M, thereby reducing the set of vertices necessary to connect. Despite this reduction in the vertex set, it turns out that weak connectivity can be established for certain pairs of vertices of G_M, as shown in the next three theorems.

Theorem 2: Let $x \in M$ with $x \equiv 5 \pmod 6$, and define $T^{-1}(x)$ to be the smallest y in $2\mathbf{N}+1$ satisfying $T(y) = x$. Then x and $T^{-1}(x)$ are weakly connected vertices of G_M.

Proof: Letting $x = 6t+5$, it follows that $T^{-1}(x) = 4t+3$. We must show that $4t+3$ is a vertex of G_M. If $4t+3$ is not in M, then there exists $w < 4t+3$ with $w \approx 4t+3$, and using the definition of $\approx$, it follows that $T(w) \approx T(4t+3) = x$. Since $x \in M$, we obtain $x \le T(w)$, and this yields

$$6t+5 \le \frac{3w+1}{2^j}, \text{ where } j \ge 1.$$

Substituting the inequality $w < 4t+3$ yields

$$6t+5 < \frac{3(4t+3)+1}{2^j} = \frac{12t+10}{2^j} = 6t+5.$$

a contradiction. Hence, $x = 4t+3$ is a vertex of G_M. Finally, since

$$\hat{T}(T^{-1}(x)) = \chi(C_{T(T^{-1}(x))}) = \chi(C_x) = x,$$

we have x and $T^{-1}(x)$ weakly connected. □

Remark: If $x \in M$ with $x \equiv 1 \pmod 6$, then $T^{-1}(x)$ is not necessarily in M. For example, $379 = \chi(L_{19})$ and $283 = \chi(L_{20})$, but $T^{-1}(379) = 505$.

Theorem 3: Let $x \in M$ with $x \equiv 1 \pmod 8$, and let $y = \chi(C_{T(x)})$. Assume y is not a multiple of 3. Then $T(x) = y$, and x and y are weakly connected vertices of G_M.

Proof: Let $x = 8k+1$. If $T(x) \ne y$, then $y = \chi(C_{T(x)})$ implies that $y < T(x)$ and $y \approx T(x)$. Also, by hypothesis, y must be of the form $6t+1$ or $6t+5$. If $y = 6t+1$, then $y < T(x)$ gives $6t+1 < 6k+1$, hence $t < k$. Also, $y \approx T(x)$ implies $T^{-1}(y) \approx x$, where $T^{-1}(y)$ is the smallest inverse image of y under T. Therefore, $8t+1 \approx x$, and since $x \in M$, we must have $x \le 8t+1$. This yields $8k+1 \le 8t+1$, hence $k \le t$, a contradiction. If y is of the form $6t+5$, then $y < T(x)$ yields $6t+5 < 6k+1$, hence $t < k$. The condition $T^{-1}(y) \approx x$ yields $4t+3 \approx x$, hence $8k+1 \le 4t+3$. Substituting the inequality $t < k$ yields $8t+1 \le 4t+3$, again a contradiction. Therefore, $T(x) = y$ must hold. Since $\hat{T}(x) = \chi(C_{T(x)}) = y$, it follows that x and y are weakly connected vertices of G_M. □

Theorem 4: Let $x \in M$ with $x \equiv 25 \pmod{64}$, and let $y = \chi(C_{T(x)})$. Then $y = [3(x-1)]/8$, and the vertices x and $[3(x-1)]/8$ are weakly connected.

Proof: Let $x = 64k+25$. Simple computations show that $T(x) = 48k+19$ and that

$$T^2\left(\frac{T(x)-1}{2}\right) = T^2(T(x)).$$

Therefore, $[T(x)-1]/2 \approx T(x)$, hence $T(x) \ne y$. Also $x \equiv 1 \pmod 8$, so we can apply Theorem 3 to see that y must be a multiple of 3. Let $y = 3t$. By Theorem 1, we have $y \approx 2y+1 = 6t+1$. Since $T(8t+1) = 6t+1$, we have $T(8t+1) \approx y$, and using the fact that $y \approx T(x)$ along with the transitivity of $\approx$, we obtain $T(8t+1) \approx T(x)$. Using the definition of $\approx$, it follows that $8t+1 \approx x$, and since $x \in M$, we have $x \le 8t+1$. Furthermore, $y \approx T(x) \approx [T(x)-1]/2 = 24k+9$; thus, by the minimality of y, we see that $3t \le 24k+9$. From this inequality, we get $\frac{8}{3}(3t)+1 \le \frac{8}{3}(24k+9)+1$ which yields $8t+1 \le x$. Therefore, $x = 8t+1$, hence $y = [3(x-1)]/8$. It follows that $\hat{T}(x) = [3(x-1)]/8$ and that x and $[3(x-1)]/8$ are weakly connected. □

Observe that, if $x > 5$ is a vertex of G_M, then, by the proof of Theorem 1, it must be true that $x \in S_1 \cup S_2$. We can actually restrict the vertex set of G_M slightly further, according to the next theorem.

Theorem 5: If x is in the arithmetic progression $\{32m+17\}_{m=1}^{\infty}$, then x is not a vertex of G_M.

Proof: We will assume $x \in M$ and find a contradiction. Let $x = 32k+17$ with $k \geq 1$, and let $y = \chi(C_{T(x)})$. Since $T(x) = 24k+13$ and $T(24k+13) = T(6k+3)$, it follows that $T(x) \neq y$. Also, $x \equiv 1 \pmod 8$ and $x \in M$, so we can apply Theorem 3 to see that y must be a multiple of 3. Now, $y \approx 2y+1$ by Theorem 1 and $y \approx T(x)$ by the definition of y, hence $T(x) \approx 2y+1$. Since y is a multiple of 3, $T^{-1}(2y+1) = \frac{8}{3}y+1$, we have $x \approx \frac{8}{3}y+1$. This yields $x \leq \frac{8}{3}y+1$, and since $y \approx T(x) \approx 6k+3$, we have $y \leq 6k+3$. Combining these inequalities yields $x \leq 16k+9$, a contradiction. Therefore, $x \notin M$, and x is not a vertex of G_M. □

Remarks: A further systematic reduction of the vertex set beyond that of Theorem 5 would be of interest, as would further development of the weak connectivity results given in Theorems 1-4. It would also be interesting to state conditions which, when combined with the theorems in this section, would be sufficient to guarantee weak connectivity of G_M; in fact, $17 \in M$.

3. A DIFFERENT REDUCTION OF THE VERTEX SET

We now reduce the vertex set of the Collatz graph to a set properly containing M, and use this set to construct a new directed graph for which weak connectedness is equivalent to truth of the $3x+1$ Conjecture. First, we need some preliminaries to help describe our vertex set. If we let $f(x) = 4x+1$ and $g(x) = 2x+1$ and let M b defined as in Section 2, we have the following lemma.

Lemma 1: Let $x \in M$, $n \in \mathbf{N}$, and $\delta \in \{0,1\}$. Then $f^n(x) \in C_x$ when $x > 1$, and $f^n g^{\delta}(x) \in C_x$ when $x > 5$.

Proof: When $x > 1$, a quick computation shows that $T(f(x)) = T(x)$, thus $T(f^n(x)) = T(x)$, and hence $f^n(x) \in C_x$ for $n \in \mathbf{N}$. When $x > 5$, we can apply Theorem 1 to obtain $g^{\delta}(x) \in C_x$, so $f^n g^{\delta}(x) \in C_x$. □

For $x > 5$, let $G_x = \{f^n g^{\delta}(x) | n \in \mathbf{N}, \delta \in \{0,1\}\}$; for $x = 3$ and $x = 5$, let $G_x = \{f^n(x) | n \in N\}$. Note that G_x consists of a collection of vertices for which weak connectedness to x in the Collatz graph has been established. For convenience, set $G_1 = \{1\}$. Lemma 1 implies that $G_x \subseteq C_x$; therefore, it makes sense to study the vertices of C_x apart from G_x. We do so using the following inductive definition.

Definition: For $j \in N$, the j^{th} *exceptional number* in C_x is the smallest positive integer x_j satisfying $x_j \in C_x - \bigcup_{i=0}^{j} G_{x_{i-1}}$, where $G_{x_{-1}} = \emptyset$.

To clarify the previous definition, consider the example

$$G_{25} = \{25, 101, 405, \ldots\} \cup \{51, 205, 821, \ldots\}.$$

Since $\sigma(25) = 7$, it follows that $C_{25} = \{x \in \mathbf{P} | \sigma(x) = 7\}$. Direct computation shows that 217 is the smallest positive integer in $C_{25} - G_{25}$, hence 217 is the first exceptional number in C_7. Repeating the process, we compute

$$G_{217} = \{217, 869, 3477, \ldots\} \cup \{435, 1741, 6965, \ldots\},$$

and hence can verify that 433 is the smallest positive integer in $C_{25}-\{G_{25}\cup G_{217}\}$. Therefore, 433 is the second exceptional number in C_{25}. A table of exceptional numbers satisfying $\sigma(x)\le 10$ and $j\le 4$ is provided below.

TABLE 1. Exceptional Numbers ($\sigma(x)\le 10, j\le 4$)

$\sigma(x)$	x
0	1
1	5
2	3, 113, 7281, 466033, 29826161
3	17, 75, 1137, 2417, 4849
4	11, 201, 369, 401, 753
5	7, 241, 267, 497, 537
6	9, 81, 321, 331, 625
7	25, 49, 217, 433, 441
8	33, 65, 273, 289, 529
9	43, 89, 177, 385, 423
10	57, 59, 465, 473, 507

Let E denote the set of all exceptional numbers $2\mathbf{N}+1$. Using the methods of the proof of Theorem 4 of [1], it can be shown that, for $\sigma(x)>1$, $C_x-\bigcup_{i=0}^{j}G_{x_{i-1}}=\emptyset$ for all $j\in N$, hence, in this case, each $j\ge 0$ gives a distinct element of E. Furthermore, the following lemma gives a complete description of the set E.

Lemma 2: $S_1\cup S_2\cup\{3,5\}=E$.

Proof: The fact that $E\subseteq S_1\cup S_2\cup\{3,5\}$ is an immediate consequence of Lemmas 6 and 7 of [1] and Theorem 1. Therefore, we will show that $S_1\cup S_2\cup\{3,5\}\subseteq E$. If $x\in S_1\cup S_2\cup\{3,5\}$ with $x\le 11$, numerical computation shows that $x\in E$, thus we will show that, if $x>11$, then $x\in S_1\cup S_2$ gives $x\in E$. If $x\notin E$, then $x=f^n g^\delta(y)$ for some $y\in E$. If $n\ge 1$, then $x\in 8m+5$, which is impossible, hence $n=0$. Therefore, $x=g^\delta(y)$ for some $y\in E$. If $\delta=0$, we obtain $x=y$, which contradicts the fact that $x\notin E$. Hence $\delta=1$; thus $x=2y+1$. Since $x\in S_1\cup S_2$, this yields $y\in S_3\cup S_4$ with $y>5$, which contradicts the fact that $y\in E\subseteq S_1\cup S_2\cup\{3,5\}$. Hence, our assumption that $x\notin E$ must be false. □

Remark: Using the equivalence classes defined in Section 1, the proofs of Theorems 3 and 4 of [1] can immediately be generalized to the case where $\sigma(x)\le\infty$.

The primary purpose of Lemma 2 is to establish weak connectivity between certain vertices of a new directed graph (see Theorem 7). However, it is interesting to note that we can use Lemma 2 and the proof of Theorem 1 to immediately establish the following theorem, which is also given in [1].

Theorem 6: Let $x\in E$ with $x>5$. Then there exists $k\in\mathbf{N}$ such that $T^k(x)=T^k(2x+1)$.

We now use the sets G_x to construct a new partition of the positive odd integers. This partition will enable us to define a new directed graph.

Lemma 3: Let $\mathcal{P}=\{G_x\mid x\in E\}$. Then $\mathcal{P}$ is a partition of $2\mathbf{N}+1$.

Proof: Since

$$\bigcup_{x\in M} C_x = 2\mathbf{N}+1 \quad \text{and} \quad C_x = \bigcup_{x\in C_x\cap E} G_x,$$

it follows immediately that

$$\bigcup_{x\in E} G_x = 2\mathbf{N}+1.$$

It remains to show that, if x and y are in E, then $G_x \cap G_y = \emptyset$ when $G_x \neq G_y$. We will prove this by contradiction. If $z \in G_x \cap G_y$, then

$$z = f^{n_1} g^{\delta_1}(x) = f^{n_2} g^{\delta_2}(y),$$

where $n_1, n_2 \in N$, $\delta_1, \delta_2 \in \{0,1\}$, $f(x)=4x+1$, and $g(x)=2x+1$. Without loss of generality, we can consider three cases: $\delta_1=\delta_2=0$; $\delta_1=0$ and $\delta_2=1$; and $\delta_1=\delta_2=1$.

In the first case, we have

$$f^{n_1}(x) = f^{n_2}(y),$$

and since we can assume $n_1 \le n_2$ without loss of generality, we obtain $x = f^{n_2-n_1}(y)$. Assume $x \neq 5$, as the theorem follows trivially in this case. If $n_2-n_1=0$, then $G_x=G_y$ is a contradiction, and if $n_2-n_1>0$, we have x of the form $8m+5$ with $m>1$ and $x\in E$, which contradicts Lemma 2. Hence, in any event, $\delta_1=\delta_2=0$ is impossible.

In the second case, we have

$$f^{n_1}(x) = f^{n_2}(2y+1).$$

If $n_1=n_2$, then $x=2y+1$, hence $x\in G_y$. Since Theorem 6 implies that $x \in C_y$, we have contradicted the fact that $x\in E$. If $n_1<n_2$, then $x=f^{n_2-n_1}(2y=1)$; therefore, x is of the form $8m+5$ with $m\ge 1$. Since $x\in E$, we have again contradicted Lemma 2. If $n_2<n_1$, we obtain $2y+1=$ $f^{n_1-n_2}(x)$, which implies that $2y+1$ is of the form $8m+5$, contradicting the fact that y is odd. Hence, in any event, $\delta_1=0$ and $\delta_2=1$ is impossible.

Finally, the third case gives

$$f^{n_1}(2x+1) = f^{n_2}(2y+1).$$

Again, without loss of generality, assume $n_1\le n_2$. If $n_1=n_2$, then $x=y$, hence $G_x=G_y$ is a contradiction. If $n_1<n_2$, then $2x+1=f^{n_2-n_1}(2y+1)$ implies that $2x+1$ is of the form $8m+5$. This forces x to be even, again a contradiction. Thus, $\delta_1=\delta_2=1$ is also impossible, and hence our assumption that $G_x\cap G_y=\emptyset$ must be false. □

Using the partition $\mathcal{P}$, we define the equivalence relation $\sim$ as follows: $x\sim y$ if and only if x and y are in G_x for some $z\in E$. Denote by E_x the equivalence class under $\sim$ which contains x. For $e\in E$, define $\overline{T}: E\to E$ by $\overline{T}(e)=\chi(E_{T(e)})$. We now obtain another formulation of the $3x+1$ Conjecture.

$3x+1$ Conjecture (3rd form): The directed graph G_E with vertices E and directed edges from e to $\overline{T}(e)$ is weakly connected.

A portion of the directed graph G_E is displayed in Figure 3. The graph G_E collapses some vertices of $G_{2\mathbf{N}+1}$ whose trajectories enter E, while at the same time retaining enough vertices to permit establishing of substantial weak connectivity.

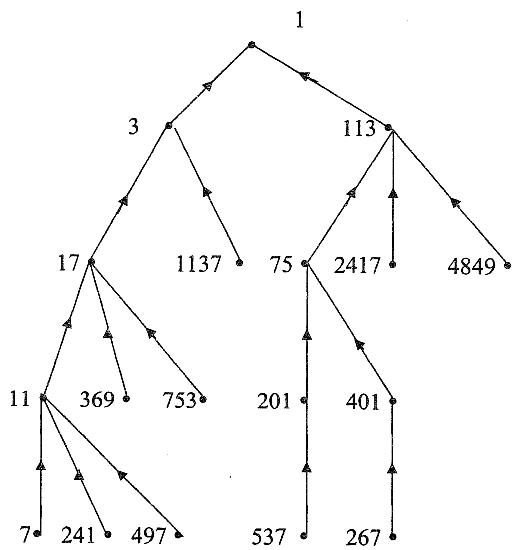

FIGURE 3. The Graph G_E $(\sigma(x) \le 5, j \le 4, x < 5000)$

Now let S_1 and S_2 be defined as in the proof of Theorem 1, and let $S = S_1 \cup S_2 - 1$. For x not a multiple of 3, let $T^{-1}(x)$ be the smallest y in $2\mathbf{N}+1$ satisfying $T(y) = x$, and define

$$T^{-1}(S) = \{T^{-1}(s) \mid s \in S - 3\mathbf{P}\}.$$

We then have the following results.

Lemma 4: $T^{-1}(S) \subseteq S$.

Proof: If $x \in S_1$, let $x = 2^{n+2}m + 2^n - 1$ with $n \in 2\mathbf{N}+1$. By considering congruences of m modulo 3, we see that x can be expressed in one of the following three forms:

$$x = 3 \cdot 2^{n+2}k + 2^n - 1;$$
$$x = 3 \cdot 2^{n+2}k + 2^{n+2} + 2^n - 1;$$
$$x = 3 \cdot 2^{n+2}k + 2^{n+3} + 2^n - 1.$$

If x is of the first form, then n odd yields $x \equiv 1 \pmod 6$. This gives

$$T^{-1}(x) = \frac{4x-1}{3}.$$

Hence $T^{-1}(x) \equiv 1 \pmod 8$, and therefore $T^{-1}(x) \in S$.

If x is of the second form, then n odd yields $x \equiv 0 \pmod 3$, hence $T^{-1}(x)$ does not exist.

If x is of the third form, then n odd yields $x \equiv 5 \pmod 6$. This gives

$$T^{-1}(x) = \frac{2x-1}{3} = 2^{n+3}k + 2^{n+2} + 2^{n+1} - 1,$$

hence $T^{-1}(x) \in S_2$. If $x \in S_2$, let $x = 2^{n+2}m + 2^{n+1} + 2^n - 1$ with $n \in 2\mathbf{P}$. Again considering congruences of m modulo 3, x can be expressed in one of the following three forms:

$$x = 3\cdot 2^{n+2}k + 2^{n+1} + 2^n - 1;$$
$$x = 3\cdot 2^{n+2}k + 2^{n+2} + 2^{n+1} + 2^n - 1;$$
$$x = 3\cdot 2^{n+2}k + 2^{n+3} + 2^{n+1} + 2^n - 1.$$

If x is of the first form, then $x \equiv 5 \pmod 6$. Therefore,

$$T^{-1}(x) = \frac{2x-1}{3} = 2^{n+3}k + 2^{n+1} - 1,$$

and hence $T^{-1}(x) \in S_1$.

If x is of the second form, then $x \equiv 0 \pmod 6$, and thus $T^{-1}(x)$ does not exist.

If x is of the third form, then $x \equiv 1 \pmod 6$ and, as before, $T^{-1}(x) \equiv 1 \pmod 8$, and thus is in S_1. Hence, in all cases, $T^{-1}(x) \in S$. □

Theorem 7: Let x be an element of E. Then the vertices x and $T^{-1}(x)$ of G_E are weakly connected.

Proof: We first show that $x \in E$ yields $T^{-1}(x) \in E$. We can assume without loss of generality that $x > 5$. Letting $x \in E$ and applying Lemma 2, we see that $x \in S_1 \cup S_2$. Applying Lemma 3 gives $T^{-1}(x) \in S_1 \cup S_2 - 1$, and again applying Lemma 2, we obtain $T^{-1}(x) \in E$. Finally, we get

$$\overline{T}(T^{-1}(x)) = \chi(E_{T(T^{-1}(x))}) = \chi(E_x) = x,$$

hence x and $T^{-1}(x)$ are weakly connected. □

4. TOTAL STOPPING TIMES OF CERTAIN EXCEPTIONAL NUMBERS AND CYCLES UNDER INDUCED MAPS

One possible approach to establishing weak connectedness of G_E is to characterize all $x \in E$ with a given finite total stopping time, and to apply T^{-1} repeatedly to those vertices. By Theorem 7, these inverse images would also be vertices of G_E, and perhaps would substantially "fill up" the set of all vertices of G_E. All $x \in E$ satisfying $\sigma(x) \le 2$ are described in Lemma 5 and Theorem 8.

Lemma 5: Let $x \in E$, and let $f(x) = 4x+1$. Then $\sigma(x) = 1$ if and only if $x = 5$.

Proof: It is well known that, for any $x \in 2\mathrm{N}+1$, $\sigma(x) = 1$ if and only if $x = \frac{1}{3}(4^{n+1}-1)$ for some $n \in P$ (see [4]). Since

$$\frac{1}{3}(4^{n+1}-1) = \sum_{i=0}^{n} 4^i = f^{n-1}(5)$$

and since $x \in E$, we must have $x = 5$. □

Lemma 6: Let

$$a_{m,n} = \frac{1}{3}\left(\sum_{i=0}^{3m} 4^{i+n} - 1\right) \quad \text{and} \quad b_{m,n} = \frac{1}{3}\left(2\sum_{i=0}^{3m-2} 4^{i+n-1} - 1\right).$$

Then $L_2 = \{a_{m,n} \mid m, n \in \mathbb{P}\} \cup \{b_{m,n} \mid m, n \in \mathbb{P}\}$.

Proof: The fact that $a_{m,n}$ and $b_{m,n}$ are in L_2 is easily verified by computation of $T^2(a_{m,n})$ and $T^2(b_{m,n})$. Thus, we need to show that $L_2 \subseteq \{a_{m,n} \mid m, n \in P\} \cup \{b_{m,n} \mid m, n \in P\}$. If $x \in L_2$, then $T(T(x))=1$, hence $T(x)=\frac{1}{3}(4^{k+1}-1)$ for some $k\in\mathbf{P}$. Since $T(x)=(3x+1)/2^j$ for some $j\in\mathbf{P}$, we obtain

$$\frac{3x+1}{2^j}=\frac{1}{3}(4^{k+1}-1)=\sum_{i=0}^{k}4^i,$$

hence $2^j\sum_{i=0}^{k}4^i\equiv 1 \pmod 3$. This yields $2^j(k+1)\equiv 1 \pmod 3$. Thus, if j is even, we have $k\equiv 0 \pmod 3$, and if j is odd, we have $k\equiv 1 \pmod 3$. In the first case, setting $j=2n$ and $k=3m$ gives $x=a_{m,n}$; in the second case, setting $j=2n-1$ and $k=3m-2$ gives $x=b_{m,n}$. □

If we let $x\in E$ with $\sigma(x)=2$, direct computation yields $E_x=\{3,113,7281,466033,\dots\}$. It is interesting to observe that the function $h(x)=64x+49$ generates all of E_x except for $x=3$, hence motivating our final lemma as well as Theorem 8.

Lemma 7: Let $x\in 2\mathbf{N}+1$, $g(x)=2x+1$, and $h(x)=64x+49$. Then $T^2(g(h^k(x)))=T^2(h^k(x))$ for all $k\in\mathbf{P}$.

Proof: We proceed by induction on k. When $k=1$, some simple computation shows that

$$T^2(g(h^k(x)))=T^2(h^k(x)).$$

Assuming the lemma is true for $k=j$, we show that the lemma holds for $k=j+1$. Since

$$T^2(g(h^{j+1}(x)))=T^2(g(h^j(h(x))))$$

and the induction hypothesis gives

$$T^2(g(h^j(h(x))))=T^2(h^j(h(x))),$$

we obtain

$$T^2(g(h^{j+1}(x)))=T^2(h^{j+1}(x)).$$

Hence, the case where $k=j+1$ holds true. □

Theorem 8: Let $x\in E$ with $x>5$ and let $h(x)=64x+49$. Then $\sigma(x)=2$ if and only if $x=h^n(1)$ for some $n\in\mathbf{P}$.

Proof: Assume $\sigma(x)=2$ and let $a_{m,n}$ and $b_{m,n}$ be defined as in Lemma 6. Using this lemma, we see that $x=a_{m,n}$ or $x=b_{m,n}$ for some $m,n\in\mathbf{P}$. If we let $f(x)=4x+1$, the relationships $a_{m,n+1}=f(a_{m,n})$ and $b_{m,n+1}=f(b_{m,n})$ are easily verified. Hence, using the fact that $x\in E$ in conjunction with Lemma 2, we see that $x=a_{m,1}$ or $x=b_{m,1}$. Now direct computation shows that $h(a_{m,1})=a_{m+1,1}$ for all $m\in\mathbf{P}$, so $a_{m+1,1}\equiv 1 \pmod 8$. Using Lemma 2 and verifying the case where $m=1$ independently, we obtain $a_{m,1}\in E$ for all $m\in\mathbf{P}$. Now let $g(x)=2x+1$. Since $T^2(a_{m,1})=T^2(b_{m+1,1})$ and $g(a_{m,1})=b_{m+1,1}$, we see that $b_{m+1,1}\in E$ only when $m=0$, hence when $b_{m+1,1}=3$. Since $x>5$, we conclude that $x=h^n(a_{m,1})$ for some $m\in\mathbf{P}$ and $n\in\mathbf{N}$. Using $h(a_{m,1})=a_{m+1,1}$ and the fact that $a_{1,1}=h(1)$, the result $x=h^n(1)$ for some $n\in\mathbf{P}$ follows.

We now show by induction that $\sigma(x)=2$ is a necessary condition for $x=h^n(1)$. For $n=1$, $\sigma(x)=2$ is easily verified. We assume that, for $x=h^n(1)\,x=h^k(1)$, we have $\sigma(x)=2$, and will show that $x=h^{k+1}(1)$ yields $\sigma(x)=2$. Direct computation shows that $T^2(h(x))=T^2(g(x))$, thus $T^2(h^{k+1}(x))=T^2(h(h^k(x)))=T^2(g(h^k(x)))$. Using Lemma 7, we obtain $T^2(h^{k+1}(x))=T^2(h^k(x))$. Finally, setting $x=1$ and invoking the induction hypothesis, we get $T^2(h^{k+1}(1))=1$; hence, for $x=h^{k+1}(1)$, we have $\sigma(x)=2$. □

Remark: A similar characterization for $x \in E$ satisfying $\sigma(x) = k$ when $k \geq 3$ would be of interest. In the case where $k = 3$, numerical computation suggests that $x = (h_1)^n(17)$ or $x = (h_2)^n(75)$, where $h_1(x) = 64x + 49$ and $h_2(x) = 32x + 17$. Furthermore, if we let $E_k = \{x \in E \mid \sigma(x) = k\}$, it can be conjectured that $E_k = \bigcup_{i=1}^{t_k} \{h_i^n(x_i) \mid n \in \mathbf{N}\}$ for some $t_k \in \mathbf{P}$ and $h_i = a_i x + b_i$. The behavior of t_k / k as $k \to \infty$ also merits further study.

We now demonstrate that a nontrivial cycle under T will induce a nontrivial cycle under the maps $\hat{T}$ and $\overline{T}$ (Theorems 9 and 10). Thus, to prove that nontrivial cycles do not exist under T, it is sufficient to prove that nontrivial cycles do not exist under either $\hat{T}$ or $\overline{T}$. Let $\approx$ and $\sim$ be the equivalence relations given in Sections 1 and 3, and let χ be defined as in Section 2. If we define $\hat{T}: 2\mathbf{N}+1 \to 2\mathbf{N}+$ by $\hat{T}(x) = \chi(C_{T(x)})$, we have the following lemmas.

Lemma 8: $\hat{T}^2(x) = \hat{T}(T(x))$ for all $x \in 2\mathbf{P}+1$.

Proof: Letting $y = \hat{T}(x)$ and $z = T(x)$, we have $y \approx z$, so $T(y) \approx T(z)$. Therefore, $C_{T(y)} = C_{T(z)}$, and thus $\chi(C_{T(y)}) = \chi(C_{T(z)})$. This gives $\hat{T}(y) = \hat{T}(z)$, and substituting for y and z gives the result. □

Lemma 9: $\hat{T}^{k+1}(x) = \hat{T}(T^k(x))$ for all $k \in \mathbf{P}$ and for all $x \in 2\mathbf{P}+1$ satisfying $\sigma(x) \geq k$.

Proof: We proceed by induction on k. The case in which $k = 1$ follows from Lemma 8. Assume that the lemma holds when $k = j$. Since

$$\hat{T}^{j+2}(x) = \hat{T}(\hat{T}^{j+1}(x)) = \hat{T}(\hat{T}(T^j(x))) = \hat{T}^2(T^j(x))$$

and since Lemma 8 gives

$$\hat{T}^2(T^j(x)) = \hat{T}(T(T^j(x))) = \hat{T}(T^{j+1}(x)),$$

the case when $k = j+1$ holds true. □

Theorem 9: If $T^k(x) = x$ for some $k \in \mathbf{P}$ and $x \in 2\mathbf{N}+1$, then there exists $y \in M$ satisfying $\hat{T}^k(y) = y$.

Proof: By Lemma 9, $\hat{T}^{k+1}(x) = \hat{T}(T^k(x))$, hence invoking the hypothesis of the theorem gives $\hat{T}^{k+1}(x) = \hat{T}(x)$, and setting $y = \hat{T}(x)$ gives the result. □

Lemma 10: Let $x, y \in E$ with $x \sim y$. Then $T(x) \sim T(y)$.

Proof: If $x \sim y$, then x and y are in G_z for some $z \in E$. Hence we can write $x = f^{n_1} g^{\delta_1}(z)$ and $y = f^{n_2} g^{\delta_2}(z)$, where $n_1, n_2 \in \mathbf{N}$, $\delta_1, \delta_2 \in \{0, 1\}$, $f(x) = 4x+1$, and $g(x) = 2x+1$. Applying Lemma 1, we see that $T(x) = T(g^{\delta_1}(x))$ and $T(y) = T(g^{\delta_2}(x))$. If $\delta_1 = \delta_2$, the result follows, so assume, without loss of generality, that $\delta_1 = 0$ and $\delta_2 = 1$. This yields $T(x) = T(z)$ and $T(y) = T(2z+1)$. If $z = 5$, the conclusion of the lemma is easily verified, so assume $z \neq 5$. Since $z \in E$, we can combine Lemma 2 with the proof of Theorem 1 to see that $T(z) = (3z+1)/2^j$ with $j = 1$ or $j = 2$. (The possibility of $j = 4$ is eliminated since $z \neq 5$.) Noting that $T(2z+1) = 3z+2$, we obtain $2^j T(z) + 1 = T(2z+1)$. When $j = 1$, this yields $g(T(x)) = T(y)$, and when $j = 2$, this yields $f(T(x)) = T(y)$; hence, in either case, $T(x) \sim T(y)$. □

Theorem 10: If $T^k(x) = x$ for some $k \in \mathbf{P}$, then there exists $e \in E$ satisfying $\overline{T}^k(e) = e$.

Proof: Using Lemma 10, the statements and proofs of Lemmas 8 and 9 hold with $\hat{T}$ replaced by $\overline{T}$, C_x replaced by E_x. and $\approx$ replaced by $\sim$. Hence, the result follows from a proof analogous to that of Theorem 9, with $\hat{T}$ replaced by $\overline{T}$ and M replaced by E. $\square$

Finally, we will demonstrate that divergent trajectories under $\hat{T}$ and $\overline{T}$ will induce divergent trajectories under T.

Theorem 11: If $\{\hat{T}^k(x)\}_{k=1}^{\infty}$ is divergent, then $\{T^k(x)\}_{k=1}^{\infty}$ is divergent.

Proof: By Lemma 9, we obtain $\hat{T}^k(x) = \hat{T}(T^{k-1}(x))$, and by the definition of $\hat{T}$, we have $\hat{T}(T^{k-1}(x)) = \chi(C_{T^k(x)})$. Thus, $\hat{T}^k(x) = \chi(C_{T^k(x)})$, and hence $\hat{T}^k(x) \le T^k(x)$, from which the theorem immediately follows. $\square$

Theorem 12: If $\{\overline{T}^k(x)\}_{k=1}^{\infty}$ is divergent, then $\{T^k(x)\}_{k=1}^{\infty}$ is divergent.

Proof: Since Lemma 9 holds with $\hat{T}$ replaced by $\overline{T}$, Theorem 12 follows from a proof analogous to that of Theorem 11, with $\hat{T}$ replaced by $\overline{T}$. $\square$

Remarks: The results in this paper are primarily geared toward a constructive proof of the $3x+1$ Conjecture by establishing weak connectivity of G_M or G_E. It is interesting to note that, if $x \equiv 1$ (mod 32) and $f(x) = 8x+9$, then x and $f(x)$ are weakly connected in G_E. Furthermore, if x is in E, $x \equiv 3$ (mod 4), and $g(x) = 32x+17$, then x and $g(x)$ are weakly connected in G_E. Finally, if $x \equiv 1$ (mod 8) and $h(x) = 64x+49$, then x and $h(x)$ are weakly connected vertices of G_E. These results, coupled with Theorems 6 and 7, may be sufficient to establish weak connectivity of G_E. This appears to be a promising direction for future research.

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