WALKING INTO AN ABSOLUTE SUM

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1. INTRODUCTION

Recently, it was asked by Paul Bruckman [1] to show that the sum

$$S_r(n) = \sum_{k=0}^{2n} \binom{2n}{k} |n-k|^r \tag{1}$$

evaluates to $n^2\binom{2n}{n}$ for $r=3$. In the published solution [16], it was also noted that $S_1(n) = n\binom{2n}{n}$, and, as a consequence, it was conjectured that $S_{2r+1}(n)$ equals the product of $\binom{2n}{n}$ and a monic polynomial of degree $r+1$.

We show this conjecture to be true, albeit with the modification of discarding the adjec-tival modifier "monic". In fact, we show that $S_{2r+1}(n) = P_r(n)n\binom{2n}{n}$ and $S_{2r}(n) = Q_r(n)2^{2n-r}$, where $P_r(n)$ and $Q_r(n)$ are both polynomials of degree r with integer coefficients. We then investigate the relationship of these polynomials to the Dumont-Foata polynomials [6]. These are generalizations of the Gandhi polynomials, which find their origin in a representation of the Genocchi numbers, first conjectured by Gandhi [9]. Finally, we show that the sums $S_r(n)$ are essentially the moments of a random variate, measuring the absolute distance to the origin in a symmetric Bernoulli random walk, after $2n$ time steps.

2. DERIVATION

We note that the sum can be rewritten as

$$S_r(n) = 2\sum_{k=0}^{n} \binom{2n}{n-k} k^r - \binom{2n}{n}\delta_{r0},$$

with δ_{r0} the Kronecker delta. Now consider, for $r \geq 1$,

$$n^2 S_r(n) - S_{r+2}(n) = 2\sum_{k=0}^{n-1} \binom{2n}{n-k} k^r (n^2 - k^2) = 4n(2n-1)\sum_{k=0}^{n-1} \binom{2n-2}{n-1-k} k^r,$$

leading directly to the recursion

$$S_{r+2}(n) = n^2 S_r(n) - 2n(2n-1)S_r(n-1). \tag{2}$$

For $r=0$, the derivation is slightly more elaborate because we need to keep track of the additional term, but leads to the same recursion so that (2) is valid for all nonnegative integers r. To start the recursion, we find the value $S_0(n) = 2^{2n}$ by an application of the binomial theorem to (1). The value of $S_1(n)$ is easily obtained by breaking up the summand k to create two sums:

$$S_1(n) = \sum_{k=0}^{n} \binom{2n}{n-k} [(n+k)-(n-k)] = 2n\sum_{k=0}^{n} \binom{2n-1}{n-k} - 2n\sum_{k=0}^{n-1} \binom{2n-1}{n-k-1},$$

and one sees that, after changing the range of summation of the second sum to start at $k=1$, all terms cancel out, with the exception of the summand $2n\binom{2n-1}{n}$. Rearranging terms gives the desired $S_1(n)=n\binom{2n}{n}$.

It is now clear that the structure of the sum depends upon the parity of r. Starting with the odd values, we simplify the recursion (2) by the substitution $S_{2r+1}(n)=P_r(n)n\binom{2n}{n}$ to give

$$P_{r+1}(n)=n^2[P_r(n)-P_r(n-1)]+nP_r(n-1), \tag{3}$$

with initial condition $P_0(n)=1$. An inductive argument now shows that $P_r(n)$ is a polynomial of degree r with integer coefficients, and proves the modified conjecture. It is not difficult to show that $r!$ is the leading coefficient of $P_r(n)$, and, hence, that these polynomials are not monic. In fact, the only cases for which the leading coefficient is 1 are $r=0$ and $r=1$. The first few polynomials are now easily determined as:

$$\begin{aligned}
&P_0(n)=1,\\
&P_1(n)=n,\\
&P_2(n)=(2n-1)n,\\
&P_3(n)=(6n^2-8n+3)n,\\
&P_4(n)=(24n^3-60n^2+54n-17)n,\\
&P_5(n)=(120n^4-480n^3+762n^2-556n+155)n.
\end{aligned}$$

For the even sums, we substitute $S_{2r}(n)=Q_r(n)2^{2n-r}$ to give the recursion

$$Q_{r+1}(n)=2n^2[Q_r(n)-Q_r(n-1)]+nQ_r(n-1), \tag{4}$$

with initial condition $Q_0(n)=1$. This shows that $Q_r(n)$ is a polynomial of degree r with integer coefficients. It is not difficult to establish that the leading coefficient is given by $(2r-1)\cdot(2r-3)\cdot \cdots 3\cdot 1=(2r)!/(2^r r!)$ and, hence, that these polynomials are also not monic. Applying the recursion gives the first few polynomials as

$$\begin{aligned}
&Q_0(n)=1,\\
&Q_1(n)=n,\\
&Q_2(n)=(3n-1)n,\\
&Q_3(n)=(15n^2-15n+4)n,\\
&Q_4(n)=(105n^3-210n^2+147n-34)n,\\
&Q_5(n)=(945n^4-3150n^3+4095n^2-2370n+496)n.
\end{aligned}$$

It is worth noting that, by evaluating $S_r(n)$ for particular values of n, one can derive various properties of [the coefficients of] the polynomials $P_r(n)$ and $Q_r(n)$. For instance, it is not difficult to show that the coefficients of $P_r(n)$ sum to unity, and those of $Q_r(n)$ to 2^{r-1} (for $r\geq 1$) by evaluating the sums for $n=1$. Indeed, one can derive the closed form solutions for $S_{2r}(n)$ and $S_{2r+1}(n)$ by solving a system of linear equations in r unknowns, representing the coefficients of the corresponding polynomial.

In the constant of the polynomials $P_r(n)/n$, one recognizes the Genocchi numbers (see [4], [10]) named after the Italian mathematician Angelo Genocchi (1817-1889):

$$G_2=-1,\ G_4=1,\ G_6=-3,\ G_8=17,\ G_{10}=-155,\ G_{12}=2073,\ldots.$$

These numbers are defined through the exponential generating function

$$\frac{2t}{e^t+1}=t+\sum_{r\geq 1} G_{2r}\frac{t^{2r}}{(2r)!},$$

and are related to the Bernoulli numbers by $G_{2r}=2(1-2^{2r})B_{2r}$. The Genocchi numbers are listed as sequence A001469 in the on-line version of the encyclopedia of integer sequences [15], where additional references may be found. The constant of the polynomials $Q_r(n)/n$ matches the first terms of the sequence A002105 in [15], and is related to the tangent numbers. The connection to the Genocchi numbers will be explored further in the next section, where the polynomials $P_r(n)$ and $Q_r(n)$ are found to be related to special cases of the Dumont-Foata polynomials.

Another matter of interest is the leading coefficient of the polynomials, characterizing the behavior of the sums $S_r(n)$ for large values of n. For the even-indexed sums, this is easily established as

$$S_{2r}(n)\sim\frac{(2r)!}{2^{2r}r!}2^{2n}n^r, \tag{5}$$

and for the odd-indexed sums we can use Stirling's formula to give $\binom{2n}{n}\sim 2^{2n}/\sqrt{\pi n}$, so that

$$S_{2r+1}(n)\sim\frac{r!}{\sqrt{\pi}}2^{2n}n^{r+\frac{1}{2}}. \tag{6}$$

In these expressions, one recognizes the moments of a central chi-distribution (see, for instance, [12], pp. 420-21). That this is no coincidence will be shown in Section 4, where we establish the connec-tion between the sums $S_r(n)$ and the distance to the origin in a symmetric Bernoulli random walk.

3. DUMONT-FOATA POLYNOMIALS

In this section we show that the polynomials $P_r(n)$ and $Q_r(n)$ are related to special cases of the Dumont-Foata polynomials [6]. These are defined recursively by means of

$$F_{r+1}(x,y,z)=(x+z)(y+z)F_r(x,y,z+1)-z^2F_r(x,y,z), \tag{7}$$

with initial condition $F_1(x,y,z)=1$. Explicit expressions for these polynomials and their generating functions have been derived by Carlitz [3], but are too lengthy to display here.

The Dumont-Foata polynomials can be regarded as generalizations of the Gandhi polynomials (see, for instance [5], [17]), which are defined by the recursion

$$F_{r+1}(z)=(z+1)^2F_r(z+1)-z^2F_r(z), \tag{8}$$

with initial condition $F_1(z)=1$. The coefficients of the first few of these polynomials are shown in Table 1, and can also be found in sequence A036970 in [15]. The Gandhi polynomials arose from a conjecture by Gandhi [9] concerning a representation of the Genocchi numbers. Gandhi's conjec-ture that $F_r(0)=(-1)^rG_{2r}$ was proved by Carlitz [2] and also by Riordan and Stein [14]. Another polynomial that can be derived as a special case of the Dumont-Foata polynomials is obtained by the recursion

$$\widetilde{F}_{r+1}(z)=(2z+1)(z+1)\widetilde{F}_r(z+1)-2z^2\widetilde{F}_r(z), \tag{9}$$

with initial condition $\widetilde{F}_1(z)=1$. The coefficients of the first few of these polynomials are given in Table 2. Comparing these and the coefficients of the Gandhi polynomials to the coefficients of the polynomials $P_r(n)$ and $Q_r(n)$, the connection to the Dumont-Foata polynomials becomes evident. By substitution in (3) and (4), it is easily verified that

$$P_r(n) = (-1)^{r-1} n F_r(1,1,-n) \text{ and } Q_r(n) = (-2)^{r-1} n F_r(\tfrac{1}{2},1,-n),$$

for $r \geq 1$. The occurrence of the Genocchi numbers in the expressions for $P_r(n)$ is seen to be a direct consequence of Gandhi's conjecture: $P_r'(0) = (-1)^{r-1} F_r(0) = -G_{2r}$. The occurrence of the Genocchi numbers in the expressions for $Q_r(n)$ is conjectured by the present author, in the form $\tilde{F}_r(0) = (-2)^r G_{2r} / (2r)$, where $\tilde{F}_r(z)$ are the polynomials defined by (9).

TABLE 1. Coefficients of the Gandhi Polynomials, Arranged in Triangular Form

							1
						2	1
					6	8	3
				24	60	54	17
			120	480	762	556	155
		720	4200	10248	12840	8146	2073
	5040	40320	139440	263040	282078	161424	38227
40320	423360	1965600	5170800	8240952	7886580	4163438	929569

TABLE 2. Coefficients of the Polynomials $\tilde{F}_r(z)$, Arranged in Triangular Form

							1
						3	1
					15	15	4
				105	210	147	34
			945	3150	4095	2370	496
		10395	51975	107415	111705	56958	11056
	135135	945945	2837835	4579575	4114110	1911000	349504
2027025	18918900	77567490	178378200	244909665	197722980	85389132	14873104

4. SYMMETRIC BERNOULLI RANDOM WALKS

In a symmetric Bernoulli random walk, one considers the movements of a particle starting at time $t = 0$ at the origin. Its movements are determined by a chance mechanism, where a fair coin is flipped and the particle is moved one unit to the right if it is heads up, and one unit to the left if it is tails up. A more exhaustive description and in-depth study of random walks can be found in Feller [8] or Révész [13]. A more playful introduction to the topic is given in the monograph by Dynkin and Uspenskii [7]. A topic of interest is the position of the particle after $2n$ coin tosses: $Y_{2n} = X_1 + X_2 + \cdots + X_{2n}$, where X_i is +1 or −1 depending upon whether or not the coin showed heads in the i^{th} coin toss. Note that the X_i are independent and identically distributed variates with mean 0 and variance 1. The probability distribution of the position of the particle after $2n$

moves can be derived from a simple combinatorial argument [see, e.g., [8], p. 75, or [13], p. 13) and is given by

$$\operatorname{Prob}(Y_{2n}=2k)=\binom{2n}{n-k}2^{-2n},$$

where $k=-n,-n+1,\ldots,n$ and n is a positive integer. The matter of interest in the context of this note is the distance to the origin $|Y_{2n}|$ at time $t=2n$. Its moments are given by

$$\mathrm{E}|Y_{2n}|^r=\sum_{k=-n}^{n}\binom{2n}{n-k}2^{-2n}|2k|^r,$$

and one sees that $\mathrm{E}|Y_{2n}|^r=2^{r-2n}S_r(n)$, thus establishing the connection to the absolute sums from the introduction. The limit behavior of these sums now becomes clear. By the central limit theorem (see, e.g., [11], p. 18), one has that Y_{2n}, for sufficiently large n, follows a normal distribution with mean 0 and variance $2n$. This implies that, asymptotically, $|Y_{2n}|$ has a half-normal or central chi-distribution, so that

$$\mathrm{E}|Y_{2n}|^r\sim\frac{\Gamma[(r+1)/2]}{\Gamma(1/2)}2^rn^{r/2}$$

(see, e.g., [12], pp. 420-21). This gives the asymptotic behavior of the sums as

$$S_r(n)=2^{2n-r}\,\mathrm{E}|Y_{2n}|^r\sim\frac{\Gamma[(r+1)/2]}{\Gamma(1/2)}2^{2n}n^{r/2},$$

and, upon expanding the gamma functions, one recovers the limit results (5) and (6).

5. DISCUSSION

One could possibly use the relation of the Gandhi polynomials to the sums $S_{2r+1}(n)$ to gain new insights on the former. In particular, one now has an expression to derive the function values of the Gandhi polynomials for negative integral arguments:

$$F_r(-n)=(-1)^{r-1}\frac{2}{n^2}\binom{2n}{n}^{-1}\sum_{k=1}^{n}\binom{2n}{n-k}k^{2r+1}\quad(n,r\geq 1).$$

For example, one easily obtains $F_r(-1)=(-1)^{r-1}$ and $F_r(-2)=(-1)^{r-1}(2^{2r-1}+1)/3$.

Likewise, one can use the relation of the moments of the absolute distance to the origin in a symmetric Bernoulli random walk and the sums $S_r(n)$ to express these moments in terms of the polynomi-als $P_r(n)$ and $Q_r(n)$:

$$\mathrm{E}|Y_{2n}|^{2r}=2^rQ_r(n)\quad\text{and}\quad\mathrm{E}|Y_{2n}|^{2r+1}=\binom{2n}{n}2^{2(r-n)+1}nP_r(n).$$

This equivalence can be used to establish the rate of convergence to the moments of the half-normal distribution.

Finally, it should be noted that one can also determine expressions for $S_{2r}(n)$ by means of the generating function

$$f_n(\varphi)=\sum_{k=0}^{2n}\binom{2n}{k}e^{(n-k)\varphi}=e^{n\varphi}[1+e^{-\varphi}]^{2n}=[e^{\varphi}+2+e^{-\varphi}]^n,$$

so that $S_{2r}(n) = f_n^{(2r)}(0)$. However, this approach covers only the even-indexed case, and does not give the same insights as the one we have followed here.

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