VIETA POLYNOMIALS

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1. VIETA ARRAYS AND POLYNOMIALS

Vieta Arrays

Consider the combinatorial forms

$$B(n, j) = \binom{n-j-1}{j} \quad \left(0 \le j \le \left[\tfrac{n-1}{2}\right]\right) \tag{1.1}$$

and

$$b(n, j) = \frac{n}{n-j}\binom{n-j}{j} \quad \left(0 \le j \le \left[\tfrac{n}{2}\right]\right), \tag{1.2}$$

where $n(\ge 1)$ is the n^{th} row in an infinite left-adjusted triangular array. Then the entries in these arrays are as exhibited in Tables 1 and 2.

TABLE 1. Array for $B(n, j)$

1				
1				
1	1			
1	2			
1	3	1		
1	4	3		
1	5	6	1	
1	6	10	4	
1	7	15	10	1
1	8	21	20	5
⋮	⋮	⋮	⋮	⋮

TABLE 2. Array for $b(n, j)$

1					
1	2				
1	3				
1	4	2			
1	5	5			
1	6	9	2		
1	7	14	7		
1	8	20	16	2	
1	9	27	30	9	
1	10	35	50	25	2
⋮	⋮	⋮	⋮	⋮	⋮

In the notation and nomenclature of this paper, Table 1 will be called the *Vieta-Fibonacci array* and Table 2 the *Vieta-Lucas array*. The Table 2 array has already been displayed in [5] where its discovery is attributed to Vieta (or Viète, 1540-1603) [8].

Vieta Polynomials

From (1.1) and Table 1, we define the *Vieta-Fibonacci polynomials* $V_n(x)$ by

$$V_n(x) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n-k-1}{k} x^{n-2k-1}, \; V_0(x) = 0. \tag{1.3}$$

From (1.3), we find:

$$\left.\begin{array}{l} V_1(x) = 1,\ V_2(x) = x,\ V_3(x) = x^2 - 1,\ V_4(x) = x^3 - 2x, \\ V_5(x) = x^4 - 3x^2 + 1,\ V_6(x) = x^5 - 4x^3 + 3x,\ V_7(x) = x^6 - 5x^4 + 6x^2 - 1, \ldots \end{array}\right\} \tag{1.4}$$

Equation (1.2) and Table 2 then invite the definition of the *Vieta-Lucas polynomials* $v_n(x)$ as

$$v_n(x)=\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\frac{n}{n-k}\binom{n-k}{k}x^{n-2k},\ v_0(x)=2. \tag{1.5}$$

From (1.5), we get:

$$\left.\begin{array}{l} v_1(x)=x,\ v_2(x)=x^2-2,\ v_3(x)=x^3-3x,\ v_4(x)=x^4-4x^2+2, \\ v_5(x)=x^5-5x^3+5x,\ v_6(x)=x^6-6x^4+9x^2-2,\dots \end{array}\right\} \tag{1.6}$$

Remark: Array Table 2 [8] and polynomials $v_n(x)$ were investigated in some detail in [5], while some fruitful pioneer work on $v_n(x)$ was accomplished in [3]. Array Table 1 and polynomials $V_n(x)$ were introduced in [6]. But see also [1, p. 14] and [4, pp. 312-13].

Recurrence Relations

Recursive definitions of the Vieta polynomials are

$$V_n(x)=xV_{n-1}(x)-V_{n-2}(x) \tag{1.7}$$

with

$$V_0(x)=0,\ V_1(x)=1, \tag{1.7a}$$

and

$$v_n(x)=xv_{n-1}(x)-v_{n-2}(x) \tag{1.8}$$

with

$$v_0(x)=2,\ v_1(x)=x. \tag{1.8a}$$

Characteristic Equation Roots

Both (1.7) and (1.8) have the characteristic equation

$$\lambda^2-\lambda x+1=0 \tag{1.9}$$

with roots

$$\alpha=\frac{x+\Delta}{2},\ \beta=\frac{x-\Delta}{2},\ \Delta=\sqrt{x^2-4} \tag{1.10}$$

so that

$$\alpha\beta=1,\ \alpha+\beta=x. \tag{1.11}$$

Purpose of this Paper

It is proposed

(i) to develop salient properties of $V_n(x)$ and $v_n(x)$, and

(ii) to explore the interplay of relationships among Vieta, Jacobsthal, and Morgan-Voyce polynomials (while observing the known connections with Fibonacci, Lucas, and Chebyshev polynomials).

2. VIETA-FIBONACCI POLYNOMIALS $V_n(x)$

Formulas (2.1) and (2.2) below flow from routine processes.

Binet Form

$$V_n(x)=\frac{\alpha^n-\beta^n}{\Delta}. \tag{2.1}$$

Generating Function

$$\sum_{n=1}^{\infty}V_n(x)y^{n-1}=[1-xy+y^2]^{-1}. \tag{2.2}$$

Simson's Formula

$$V_{n+1}(x)V_{n-1}(x) = V_n^2(x) = -1 \quad \text{(by (2.1))]}. \tag{2.3}$$

Negative Subscript

$$V_{-n}(x) = -V_n(x) \quad \text{(by (2.1))}. \tag{2.4}$$

Differentiation

$$\frac{dv_n(x)}{dx} = nV_n(x) \quad \text{(by (2.1), (3.1))}. \tag{2.5}$$

A neat result:

$$V_n(x)V_{n-1}(-x) + V_n(-x)V_{n-1}(x) = 0 \quad (n \geq 2). \tag{2.6}$$

Induction may be used to demonstrate (2.6); see [6].

3. VIETA-LUCAS POLYNOMIALS $v_n(x)$

Standard techniques reveal the following basic features of $v_n(x)$.

Binet Form

$$v_n(x) = \alpha^n + \beta^n. \tag{3.1}$$

Generating Function

$$\sum_{n=0}^{\infty} v_n(x)y^n = (2 - xy)[1 - xy + y^2]^{-1}. \tag{3.2}$$

Simson's Formula

$$v_{n+1}(x)v_{n-1}(x) - v_n^2(x) = \begin{cases} -1 & n \text{ odd,} \\ \Delta^2 & n \text{ even.} \end{cases} \tag{3.3}$$

Negative Subscript

$$v_{-n}(x) = v_n(x). \tag{3.4}$$

Miscellany

$$v_n(x)v_{n-1}(-x) + v_n(-x)v_{n-1}(x) = 0. \tag{3.5}$$

$$v_n^2(x) + v_{n-1}^2(x) - xv_n(x)v_{n-1}(x) = -\Delta^2. \tag{3.6}$$

$$v_n(x^2 - 2) - v_n^2(x) = -2. \tag{3.7}$$

Remarks:

(i) Results (3.3)-(3.7) may be determined by applying (3.1). To establish (3.5) by an alternative method, follow the approach used in [6] for the analogous equation for $V_n(x)$.

(ii) Both (3.6) and (3.7) occur, in effect, in [3].

(iii) There are no results for $V_n(x)$ corresponding to (3.6) and (3.7) for $v_n(x)$.

(iv) Observe that, for $v_n(x^2 - 2)$, the expressions corresponding to α, β, and Δ in (1.10) become $\alpha^* = \alpha^2$, $\beta^* = \beta^2$, $\Delta^* = x\Delta$.

Permutability

Theorem 1 (Jacobsthal [3]): $v_m(v_n(x)) = v_n(v_m(x)) = v_{mn}(x)$.

Proof: Adapting Jacobsthal's neat treatment of this elegant result, we notice the key nexus

$$v_n(x) = v_n\left(\alpha + \frac{1}{\alpha}\right) = \alpha^n + \alpha^{-n} \quad \text{(by (1.11), (3.1))}. \tag{3.8}$$

whence

$$\begin{aligned} v_{mn}(x) &= \alpha^{nm} + \alpha^{-nm} && \text{(by (3.1))} \\ &= v_n(\alpha^m + \alpha^{-m}) && \text{(by (3.8))} \\ &= v_n(v_m(x)) && \text{(by (3.1))} \\ &= v_m(v_n(x)) && \text{also.} \end{aligned}$$

Remark: There is no result for $V_n(x)$ corresponding to Theorem 1 (Jacobsthal's theorem) for $v_n(x)$, i.e., the $V_n(x)$ are nonpermutable [cf. (9.3), (9.4)].

4. PROPERTIES OF $V_n(x)$, $v_n(x)$

Elementary methods, mostly involving Binet forms (2.1) and (3.1), disclose the following quintessential relations connecting $V_n(x)$ and $v_n(x)$.

$$V_n(x)v_n(x) = V_{2n}(x). \tag{4.1}$$

$$V_{n+1}(x) - V_{n-1}(x) = v_n(x). \tag{4.2}$$

$$v_{n+1}(x) - v_{n-1}(x) = \Delta^2 V_n(x). \tag{4.3}$$

$$v_n(x) = 2V_{n+1}(x) - xV_n(x). \tag{4.4}$$

$$\Delta^2 V_n(x) = 2v_{n+1}(x) - xv_n(x). \tag{4.5}$$

Notice that (4.4) is a direct consequence of the generating function definitions (2.2) and (3.2).

Summation

$$\Delta^2 \sum_{n=1}^{m} V_n(x) = v_{m+1}(x) + v_m(x) - x - 2 \quad \text{(by (4.3))}. \tag{4.6}$$

$$\sum_{n=1}^{m} v_n(x) = V_{m+1}(x) + V_m(x) - 1 \quad \text{(by (4.2))}. \tag{4.7}$$

Sums (Differences) of Products

$$V_m(x)v_n(x) + V_n(x)v_m(x) = 2V_{m+n}(x). \tag{4.8}$$

$$V_m(x)v_n(x) - V_n(x)v_m(x) = 2V_{m-n}(x). \tag{4.9}$$

$$v_m(x)v_n(x) + \Delta^2 V_m(x)V_n(x) = 2v_{m+n}(x). \tag{4.10}$$

$$v_m(x)v_n(x) - \Delta^2 V_m(x)V_n(x) = 2v_{m-n}(x). \tag{4.11}$$

Special cases $m = n$: In turn, the reductions are (4.1), $0 = 0$ (1.7a), and

$$v_n^2(x) + \Delta^2 V_n^2(x) = 2v_{2n}(x) \quad \text{(by (4.10))}, \tag{4.12}$$

$$v_n^2(x) - \Delta^2 V_n^2(x) = 4 \quad \text{(by (4.11))}. \tag{4.13}$$

Associated Sequences

Definitions: The k^{th} *associated sequences* $\{V_n^{(k)}(x)\}$ and $\{v_n^{(k)}(x)\}$ of $\{V_n(x)\}$ and $\{v_n(x)\}$ are defined by, respectively ($k \geq 1$),

$$V_n^{(k)}(x) = V_{n+1}^{(k-1)}(x) - V_{n-1}^{(k-1)}(x), \tag{4.14}$$

$$v_n^{(k)}(x) = v_{n+1}^{(k-1)}(x) - v_{n-1}^{(k-1)}(x), \tag{4.15}$$

where $V_n^{(0)}(x) = V_n(x)$ and $v_n^{(0)}(x) = v_n(x)$.

What are the ramifications of these ideas?

Immediately,

$$V_n^{(1)}(x) = v_n(x) \quad \text{(from (4.2))}, \tag{4.16}$$

$$v_n^{(1)}(x) = \Delta^2 V_n(x) \quad \text{(from (4.3))} \tag{4.17}$$

are the generic members of the *first associated sequences* $\{V_n^{(1)}(x)\}$ and $\{v_n^{(1)}(x)\}$.

Repeated application of the above formulas eventually reveals the succinct results:

$$V^{2m}(x) = v_n^{(2m-1)}(x) = \Delta^{2m} V_n(x), \tag{4.18}$$

$$V_n^{(2m+1)}(x) = v_n^{2m}(x) = \Delta^{2m} v_n(x). \tag{4.19}$$

5. THE ARGUMENT $-x^2$: VIETA AND MORGAN-VOYCE

Attractively simple formulas can be found to relate the Vieta polynomials to Morgan-Voyce polynomials having argument $-x^2$. Valuable space is preserved in this paper by asking the reader to consult [2] and [6] for the relevant combinatorial definitions of the Morgan-Voyce polynomials $B_n(x)$, $b_n(x)$, $C_n(x)$, and $c_n(x)$.

Alternative proofs are provided specifically to heighten insights into the structure of the polynomials. Equalities in some proofs require a reverse order of terms.

Theorem 2:

(a) $V_{2n}(x) = (-1)^{n-1} x B_n(-x^2)$.

(b) $V_{2n-1}(x) = (-1)^{n-1} b_n(-x^2)$.

(a)

Proof 1:

$$\begin{aligned}(-1)^{n-1} x B_n(-x^2) &= \sum_{k=0}^{n-1} (-1)^{k+n-1} \binom{n+k}{2k+1} x^{2k+1} \quad \text{(by [6, (2.20)])}\\ &= V_{2n}(x) \quad \text{(by (1.3))}.\end{aligned}$$

Proof 2:

$$\begin{aligned}V_{2n}(x) &= (-1)^{n-1} x [b_n(-x^2) + B_{n-1}(-x^2)] \quad \text{(by [6] adjusted)}\\ &= (-1)^{n-1} x B_n(-x^2) \quad \text{(by [2, (2.13)])}.\end{aligned}$$

(b)

Proof 1:

$$\begin{aligned}(-1)^{n-1} b_n(-x^2) &= \sum_{k=0}^{n-1} (-1)^{k+n-1} \binom{n+k-1}{2k} x^{2k} \quad \text{(by [2, (2.21)])}\\ &= V_{2n-1}(x) \quad \text{(by (1.13))}.\end{aligned}$$

Proof 2:

$$\begin{aligned}V_{2n-1}(x) &= (-1)^n (x^2 B_n(-x^2) - b_{n-1}(-x^2)) \quad \text{(by [6] adjusted)}\\ &= (-1)^n (-b_n(-x^2)) \quad \text{(by [2, (2.15)])}\\ &= (-1)^{n-1} b_n(-x^2).\end{aligned}$$

Corollary 1: $V_{2n-1}(ix) = (-1)^{n-1} b_n(x^2)$ $(i^2 = -1)$.

Theorem 3:

(*a*) $v_{2n}(x) = (-1)^n C_n(-x^2)$.

(*b*) $v_{2n-1}(x) = (-1)^{n-1} x c_n(-x^2)$.

(a)

Proof:

$$\begin{aligned}(-1)^n C_n(-x^2) &= (-1)^n \left\{ \sum_{k=0}^{n-1} (-1)^k \frac{2n}{n-k} \binom{n-1+k}{n-1-k} x^{2k} + (-1)^n x^{2n} \right\} \quad \text{(by [6, (2.2)])} \\ &= v_{2n}(x) \quad \text{(by (1.5))} \\ &[= (-1)^n (C_{n-1}(-x^2) - x^2 c_n(-x^2)) \quad \text{(by (3.21)])}.\end{aligned}$$

(b)

Proof:

$$\begin{aligned}(-1)^{n-1} x c_n(-x^2) &= \sum_{k=1}^{n} (-1)^{k+n} \frac{2n-1}{2k-1} \binom{n+k-2}{n-k} x^{2k-1} \quad \text{(by [2, (3.23)])} \\ &= v_{2n-1}(x) \quad \text{(by (1.5))} \\ &[= (-1)^{n-1} x (C_{n-1}(-x^2) + c_{n-1}(-x^2)) \quad \text{(by [2, (3.11)]])}.\end{aligned}$$

Corollary 2: $v_{2n}(ix) = (-1)^n C_n(x^2)$ $(i^2 = -1)$.

6. THE ARGUMENT $-\frac{1}{x^2}$: VIETA AND JACOBSTHAL

Here, we discover connections between the Vieta and Jacobsthal polynomials.

Theorem 4:

(*a*) $V_n(x) = x^{n-1} J_n\left(-\frac{1}{x^2}\right)$.

(*b*) $v_n(x) = x^n j_n\left(-\frac{1}{x^2}\right)$ (by [6, (2.7)]).

(a)

Proof:

$$\begin{aligned}V_n(x) &= x^{n-1} \sum_{j=0}^{[\frac{n-1}{2}]} \binom{n-j-1}{j} \left(-\frac{1}{x^2}\right)^j \quad \text{(by (1.3))} \\ &= x^{n-1} J_n\left(-\frac{1}{x^2}\right) \quad \text{(by [6, (2.3)])}\end{aligned}$$

$$\left[\begin{aligned}&= x^{n-1}\left[J_{n-1}\left(-\tfrac{1}{x^2}\right) + \left(-\tfrac{1}{x^2}\right) J_{n-2}\left(-\tfrac{1}{x^2}\right)\right] \text{ by definition of } J_n(x) \\ &= x^{n-1} J_{n-1}\left(-\tfrac{1}{x^2}\right) - x^{n-3} J_{n-2}\left(-\tfrac{1}{x^2}\right) \text{ as in [6] adjusted}\end{aligned}\right].$$

(b)

Proof:

$$x^n j_n\left(-\frac{1}{x^2}\right)=\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\frac{n}{n-k}\binom{n-k}{k}x^{n-2k}\quad \text{(by [6, (2.6)])}$$
$$=v_n(x)\quad \text{(by (1.5) or [5, (1.9)])}$$
$$\left[\begin{array}{l} =x^n\left[j_{n-1}\left(-\tfrac{1}{x^2}\right)+\left(-\tfrac{1}{x^2}\right)j_{n-2}\left(-\tfrac{1}{x^2}\right)\right] \text{ by definition of } j_n(x) \\ =x^n j_{n-1}\left(-\tfrac{1}{x^2}\right)-x^{n-2}j_{n-2}\left(-\tfrac{1}{x^2}\right)\end{array}\right].$$

7. THE ARGUMENT $\frac{1}{x}$: JACOBSTHAL AND MORGAN-VOYCE

Next, we detect some attractive simple links between Jacobsthal and Morgan-Voyce polynomials involving reciprocal arguments x, $\frac{1}{x}$.

Theorem 5:

(a) $B_n(x)=x^{n-1}J_{2n}\left(\frac{1}{x}\right)$.

(b) $C_n(x)=x^n j_{2n}\left(\frac{1}{x}\right)$.

(a) This is stated and proved in [6, (2.8)].

(b)

Proof:

$$x^n j_{2n}\left(\frac{1}{x}\right)=\sum_{k=0}^{n}\frac{2n}{2n-k}\binom{2n-k}{k}x^{n-k}\quad \text{(by [6, (2.6)])}$$
$$=\sum_{k=0}^{n-1}\frac{2n}{2n-k}\binom{2n-k}{k}x^{n-k}+2$$
$$=C_n(x)\quad \text{(by [6, (2.2)])}.$$

Upon making the transformation $x\rightarrow\frac{1}{x}$ in Theorem 5(a) and (b), we obtain their *Mutuality Properties* in Corollary 3(a) and (b).

Corollary 3 (Mutuality):

(a) $J_{2n}(x)=x^{n-1}B_n\left(\frac{1}{x}\right)$.

(b) $j_{2n}(x)=x^n C_n\left(\frac{1}{x}\right)$.

Combining Theorems 2(a) and 4(a), we get

$$x^{2n-1}J_{2n}\left(-\frac{1}{x^2}\right)=V_{2n}(x)=(-1)^{n-1}xB_n(-x^2)$$

leading to

$$B_n(-x^2) = (-x^2)^{n-1} J_{2n}\left(-\frac{1}{x^2}\right),$$

thus confirming Theorem 5(a) when $x \to -x^2$. Conclusions of a similar nature link $j_{2n}(-\frac{1}{x^2})$, $v_{2n}(x)$, and $b_n(-x^2)$ in Theorems 3(a), 4(b), and 5(b).

Theorem 6:

(a) $b_n(x) = x^{n-1} J_{2n-1}\left(\frac{1}{x}\right)$.

(b) $c_n(x) = x^{n-1} j_{2n-1}\left(\frac{1}{x}\right)$.

Proof: Similar to that for Theorem 5.

Corollary 4 (Mutuality):

(a) $J_{2n-1}(x) = x^{n-1} b_n\left(\frac{1}{x}\right)$.

(b) $j_{2n-1}(x) = x^{n-1} c_n\left(\frac{1}{x}\right)$.

8. ZEROS OF $V_n(x)$, $v_n(x)$

Known zeros of the Morgan-Voyce polynomials [2, (4.20)-(4.23)] may be employed to detect the zeros of the Vieta and the Jacobsthal polynomials. Some elementary trigonometry is required.

(a) $V_n(x) = 0$

By [2, (4.20)] and Theorem 2(a) with $x \to -x^2$, the $2n-1$ zeros of $V_{2n}(x)$ are 0 and the $2(n-1)$ zeros of $B_n(-x^2)$, namely $(r = 1, 2, \ldots, n-1)$,

$$\begin{aligned} x &= \pm 2\sin\left(\frac{r}{n}\frac{\pi}{2}\right) = \pm 2\cos\left(\frac{n-r}{2n}\pi\right) \\ &= 2\cos\frac{r}{m}\pi \;\; (m = 2n, \text{ i.e., } m \text{ even}). \end{aligned} \tag{8.1}$$

Similarly, by [2, (4.21)] and Theorem 2(b) with $x \to -x^2$, the $2n-2$ zeros of $V_{2n-1}(x)$ are the $2(n-1)$ zeros of $b_n(-x^2)$, namely $(r = 1, 2, \ldots, n-1)$,

$$\begin{aligned} x &= \pm 2\sin\left(\frac{2r-1}{2n-1}\frac{\pi}{2}\right) = \pm 2\cos\left(\frac{n-r}{2n-1}\pi\right) \\ &= 2\cos\frac{r}{m}\pi \;\; (m = 2n-1, \text{ i.e., } m \text{ odd}). \end{aligned} \tag{8.2}$$

Zeros $2\cos\frac{r}{m}\pi$ given in (8.1) and (8.2) are precisely those given in [7, (2.25)] for $y = -1$ (for $V_m(x)$) when m is even or odd. See also [7, (2.23)].

(b) $v_n(x) = 0$

Invoking Theorems 3(a) and 3(b) next in conjunction with [2, (4.22), (4.23)] for $C_n(x)$ and $c_n(x)$ and making the transformation $x \to -x^2$, we discover the n zeros of $v_n(x)$ are $(r = 1, \dots, n)$

$$x = 2\cos\left(\frac{2r-1}{2n}\pi\right)$$

which is in accord with [7, (2.26)]. See also [7, (2.24)].

Alternative approach to (a) and (b) above: Use the known roots for Chebyshev polynomials (9.3) and (9.4).

(c) Zeros of $J_n(x)$, $j_n(x)$

From Theorems 4(a), 4(b), it follows that the zeros of $J_n(x)$, $j_n(x)$ are given by $-\frac{1}{x^2} \to x$. This leads in (8.1)-(8.3) to the zeros of $J_n(x)$, $j_n(x)$ as

$$-\frac{1}{4\cos^2\frac{r\pi}{n}}, \quad -\frac{1}{4\cos^2\left(\frac{2r-1}{2n}\pi\right)},$$

that is, for

(c) $J_n(x) = 0$: $\quad x = -\frac{1}{4}\sec^2\frac{r\pi}{n}, \qquad (8.4)$

(d) $j_n(x) = 0$: $\quad x = -\frac{1}{4}\sec^2\left(\frac{2r-1}{2n}\pi\right). \qquad (8.5)$

These zero values concur with those given in [7, (2.28(, (2.29)] if we remember that $2x$ in the definitions for $J_n(x)$, $j_n(x)$ in [7] has to be replaced by x in this paper (as in [6]). Refer also to Corollaries 3(a) and 3(b).

9. MEDLEY

Lastly, we append some Vieta-related features of familiar polynomials.

Fibonacci and Lucas Polynomials $F_n(x)$, $L_n(x)$

$$V_n(ix) = i^{n-1}F_n(x) \quad (i^2 = -1). \qquad (9.1)$$

$$v_n(ix) = i^n L_n(x) \quad ([5]). \qquad (9.2)$$

Chebyshev Polynomials $T_n(x)$, $U_n(x)$

$$V_n(x) = U_n\left(\frac{1}{2}x\right). \qquad (9.3)$$

$$v_n(x) = 2T_n\left(\frac{1}{2}x\right) \quad ([3], [5]). \qquad (9.4)$$

Suggested Topics for Further Development

1. **Irreducibility, divisibility:** Detailed analysis for $v_n(x)$ as in [5] is, for $V_n(x)$, left to the *aficionados* (having regard to Tables 1 and 2);
2. **Rising and falling diagonals** for Vieta polynomials (which has already been done for the Chebyshev polynomials and which has been almost completed for Vieta polynomials);

3. **Convolutions** for $V_n(x)$ and $v_n(x)$ (in which much progress has been achieved);
4. **Numerical values:** Consider various integer values of x in $V_n(x)$ and $v_n(x)$ to obtain sets of *Vieta numbers*. Some nice results ensue. Guidance may be sought in [2, pp. 172-73].

Conclusion

Apparently the $v_n(x)$ offer a slightly richer field of exploration than do the $V_n(x)$. However, many opportunities for discovery present themselves. Hopefully, this paper may whet the appetite of some readers to undertake further experiences.

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REFERENCES

1. A. F. Horadam. "Tschebyscheff and Other Functions Associated with the Sequence $\{W_n(a, b; p, q)\}$." *The Fibonacci Quarterly* **7.1** (1969):14-22.
2. A. F. Horadam. "New Aspects of Morgan-Voyce Polynomials." In *Applications of Fibonacci Numbers* **7:**161-76. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1998.
3. E. Jacobsthal. "Ueber Vertauschbare Polynome." *Math. Zeits.* **63** (1955):244-76.
4. E. Lucas. *Théorie des Nombres*. Paris: Blanchard, 1961.
5. N. Robbins. "Vieta's Triangular Array and a Related Family of Polynomials." *Internat. J. Math. & Math. Sci.* **14.2** (1991):239-44.
6. A. G. Shannon & A. F. Horadam. "Some Relationships among Vieta, Morgan-Voyce, and Jacobsthal Polynomials." In *Applications of Fibonacci Numbers* **8:**307-23. Ed. F. Howard. Dordrecht: Kluwer, 1999.
7. M. N. S. Swamy. "Generalized Fibonacci and Lucas Polynomials and Their Associated Diagonal Polynomials." *The Fibonacci Quarterly* **37.3** (1999):213-22.
8. F. Vieta. *Opera Mathematica: Ad Angulus Sectiones.* (Theorema VI). Paris, 1615. (Quoted in [5].)

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