THE INTERVAL ASSOCIATED WITH A FIBONACCI NUMBER

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1. INTRODUCTION

It is well known that the n^{th} Fibonacci number F_n is given by the Binet-Moivre form $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, where $\alpha, \beta = (1 \pm \sqrt{5})/2$. Möbius [2], however, gave a different way to characterize a Fibonacci number. Let z be an integer with $z \ge 2$. Then z is a Fibonacci number if and only if the interval $[gz - 1/z, gz + 1/z]$ contains exactly one integer, where $g = \alpha = (1+\sqrt{5})/2$ is the golden number.

In this paper we shall give some criteria about a more general case.

2. CRITERION 1

As usual, let $\alpha = [a_0; a_1, a_2, \dots]$ denote the continued fraction expansion of α, where

$$\alpha = a_0 + 1/\alpha_1, \qquad a_0 = \lfloor \alpha \rfloor,$$
$$\alpha_n = a_n + 1/\alpha_{n+1}, \quad a_n = \lfloor \alpha_n \rfloor \quad (n = 1, 2, \dots).$$

The n^{th} convergent $p_n / q_n = [a_0; a_1, \dots, a_n]$ of α is given by the recurrence relations

$$p_n = a_n p_{n-1} + p_{n-2} \quad (n = 0, 1, \dots), \qquad p_{-2} = 0, \quad p_{-1} = 1,$$
$$q_n = a_n q_{n-1} + q_{n-2} \quad (n = 0, 1, \dots), \qquad q_{-2} = 1, \quad q_{-1} = 0.$$

Let the sequence G_n be defined by $G_0 = 0$, $G_1 = 1$, $G_n = aG_{n-1} + G_{n-2}$ $(n = 2, 3, \dots)$. G_n is called the n^{th} *generalized Fibonacci number*. The Binet-Moivre form of G_n $(n = 0, 1, 2, \dots)$ is given by

$$G_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where α and β are the solutions of the equation $x^2 - ax - 1 = 0$. Assume that $\alpha > \beta$. Then the continued fraction expansion of α is given by

$$\alpha = \frac{a + \sqrt{a^2+4}}{2} = [a; a, a, \dots]$$

and $G_n = q_{n-1} = p_{n-2}$ $(n \ge 0)$.

Theorem 1: Let $\alpha = (a + \sqrt{a^2+4})/2 = [a; a, a, \dots]$ with $a \ge 2$ (or $a = 1$ and $n \ge 2$). Then q is a generalized Fibonacci number is and only if the interval

$$\left[q\alpha - \frac{1}{aq}, \; q\alpha + \frac{1}{aq} \right]$$

contains exactly one integer p; explicitly $q = q_n = G_{n+1}$ and $p = p_n$.

Remark: In fact,

$$p_n - 1 < q_n\alpha - \frac{1}{aq_n} < p_n < q_n\alpha + \frac{1}{aq_n} < p_n + 1.$$

Proof: In general, by [3],

$$\left|\alpha-\frac{p_n}{q_n}\right|=\frac{1}{q_n(\alpha_{n+1}q_n+q_{n-1})}<\frac{1}{a_{n+1}q_n^2}$$

because $\alpha_{n+1}=[a_{n+1};a_{n+2},\dots]$. When $\alpha=(\sqrt{5}+1)/2=[1;1,1,\dots]$ and $n\ge 2$, a more precise upper bound is possible. Namely, from

$$\alpha_{n+1}+\frac{q_{n-1}}{q_n}\ge\frac{\sqrt{5}+1}{2}+\frac{1}{2}>2,$$

we have $|\alpha-p_n/q_n|<1/(2q_n^2)$.

Returning to $\alpha=(a+\sqrt{a^2+4})/2=[a;a,a,\dots]$, if $q=q_n$ $(=G_{n+1})$ and $p=p_n$, then $|\alpha-p/q|$ $<1/(aq^2)$, which is equivalent to

$$q\alpha-\frac{1}{aq}<p<q\alpha+\frac{1}{aq}.$$

Furthermore,

$$p_n+1>q_n\alpha+1-\frac{1}{aq_n}\ge q_n\alpha+\frac{1}{aq_n}\quad\text{and}\quad p_n-1<q_n\alpha-1+\frac{1}{aq_n}\le q_n\alpha-\frac{1}{aq_n}.$$

Notice that when $a=1$ and $n=0,1$, the interval contains two integers. In fact,

$$q_n\alpha-\frac{1}{q_n}=\alpha-1<1<2<\alpha+1=q_n\alpha+\frac{1}{q_n}.$$

On the other hand, suppose that p/q satisfies $|\alpha-p/q|<1/(aq^2)$. We shall follow a similar step to the proof of Theorem 184 in [1]. Assume that $p/q=[b_0;b_1,\dots,b_n]$. Then

$$\alpha-\frac{p}{q}=\frac{\varepsilon(-1)^n}{q^2}\quad\left(0<\varepsilon<\frac{1}{a}\right).$$

Set

$$\omega=\frac{P_{n-1}-\alpha Q_{n-1}}{\alpha Q_n-P_n},\ \text{ i.e., }\ \alpha=\frac{\omega P_n+P_{n-1}}{\omega Q_n+Q_{n-1}},$$

where $P_n/Q_n=p/q=[b_0;b_1,\dots,b_n]$. Then

$$\frac{\varepsilon(-1)^n}{q^2}=\alpha-\frac{P_n}{Q_n}=\frac{(-1)^n}{Q_n(\omega Q_n+Q_{n-1})}.$$

Letting $\varepsilon=Q_n/(\omega Q_n+Q_{n-1})$, we have

$$\omega=\frac{1}{\varepsilon}-\frac{Q_{n-1}}{Q_n}>a-1\ge 1\ (a\ge 2).$$

Notice again that we can set $a=2$ instead of $a=1$ when $\alpha=(\sqrt{5}+1)/2=[1;1,1,\dots]$ and $n\ge 2$. Therefore, by Theorem 172 in [1], P_{n-1}/Q_{n-1} and P_n/Q_n are two consecutive convergents to α.

3. CRITERION 2

As Möbius proved, unless α is the golden number, the number of integers included in the interval $\left[q\alpha-\frac{1}{q},q\alpha+\frac{1}{q}\right]$ may be more than one. For the generalized α, the following criterion holds.

Theorem 2: Let $\alpha = (a+\sqrt{a^2+4})/2 = [a;a,a,\ldots]$. Then the solutions (p,q) of the inequality $q\alpha - 1/q < p < q\alpha + 1/q$ using positive integers p and q are as follows:

$$(p_n, q_n), \ldots, (tp_n, tq_n),\ (p_n + p_{n-1}, q_n + q_{n-1}),\ (p_{n+1} - p_n, q_{n+1} - q_n)\ \ (n \ge 0),$$

where $t = \lfloor \sqrt{a} \rfloor$.

Proof: Let q be an integer with $q_n \le q < q_{n+1}$. First, we will show that if $|q\alpha - p| \le 1/q$ then the form of q must be iq_n or $q_{n+1} - iq_n$ $(i = 1, 2, \ldots, a_{n+1} - 1)$. By Lemma 2.1 and Theorem 3.3 in [4], we have

$$\{u_1\alpha\} < \{u_2\alpha\} < \cdots < \{u_{q_{n+1}-1}\alpha\} \quad \text{if } n \text{ is even,}$$
$$\{u_1\alpha\} > \{u_2\alpha\} > \cdots > \{u_{q_{n+1}-1}\alpha\} \quad \text{if } n \text{ is odd,}$$

where $\{u_1, u_2, \ldots, u_{q_{n+1}-1}\} = \{1, 2, \ldots, q_{n+1} - 1\}$ is a set with $u_j \equiv jq_n \pmod{q_{n+1}}$ $(j = 1, 2, \ldots, q_{n+1} - 1)$. Explicitly,

$$\begin{aligned}\{q_n\alpha\} &< \{2q_n\alpha\} < \cdots < \{a_{n+1}q_n\alpha\} < \{(q_n - q_{n-1})\alpha\} < \{(2q_n - q_{n-1})\alpha\} \\ &< \cdots < \{(q_{n+1} - 2q_n + q_{n-1})\alpha\} < \{(q_{n+1} - q_n + q_{n-1})\alpha\} < \{(q_{n+1} - a_{n+1}q_n)\alpha\} \\ &< \cdots < \{(q_{n+1} - 2q_n)\alpha\} < \{(q_{n+1} - q_n)\alpha\},\end{aligned}$$

if n is even; similar if n is odd. Since

$$\|(q_n - q_{n-1})\alpha\| = |(q_n - q_{n-1})\alpha - (p_n - p_{n-1})| = \frac{\alpha_{n+1} + 1}{\alpha_{n+1}q_n + q_{n-1}} \ge \frac{1}{q_n}$$

and

$$\|(q_{n+1} - q_n + q_{n-1})\alpha\| = |(q_{n+1} - q_n + q_{n-1})\alpha - (p_{n+1} - p_n + p_{n-1})| = \frac{\frac{1}{\alpha_{n+2}} + \alpha_{n+1} + 1}{\alpha_{n+1}q_n + q_{n-1}} \ge \frac{1}{q_n},$$

there does not exist a q satisfying $q\alpha - 1/q < p < q\alpha + 1/q$ unless the form of q is $q = iq_n$ or $q = q_{n+1} - iq_n$ $(i = 1, 2, \ldots, a_{n+1} - 1)$.

$$\left|\alpha - \frac{kp_n}{kq_n}\right| = \left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{q_n(\alpha_{n+1}q_n + q_{n-1})} \le \frac{1}{k^2 q_n^2}$$

holds if and only if

$$k \le \sqrt{\alpha_{n+1} + \frac{q_{n-1}}{q_n}}.$$

When $\alpha = [a; a, a, \ldots]$, we have

$$\sqrt{a} < \sqrt{\alpha} = \sqrt{\alpha_1 + \frac{q_{-1}}{q_0}} < \sqrt{\alpha_3 + \frac{q_1}{q_2}} < \cdots < \sqrt{\alpha_4 + \frac{q_2}{q_3}} < \sqrt{\alpha_2 + \frac{q_0}{q_1}} = \frac{\alpha}{\sqrt{a}}.$$

Since $\lfloor \sqrt{a} \rfloor = \lfloor \alpha/\sqrt{a} \rfloor = 1$ $(a = 1)$ and $\alpha/\sqrt{a} < \sqrt{a+1}$ $(a \ge 2)$, we obtain

$$\lfloor \sqrt{a} \rfloor = \left\lfloor \sqrt{\alpha_{n+1} + \frac{q_{n-1}}{q_n}} \right\rfloor = \left\lfloor \frac{\alpha}{\sqrt{a}} \right\rfloor.$$

Let $a_{n+1} \ge 4$. Then, since $a_{n+1} \ge i + 2 \ge i^2/(i-1)$ for $i = 2, 3, \ldots, a_{n+1} - 2$, we have $(i-1)\alpha_{n+2}a_{n+1} \ge i^2\alpha_{n+2}$, yielding $((i-1)\alpha_{n+2} + 1)q_{n+1} > (i^2\alpha_{n+2} + i + 1)q_n$. Thus,

$$\|(q_{n+1}-iq_n)\alpha\| = \frac{1}{\alpha_{n+2}q_{n+1}+q_n} + \frac{i}{\alpha_{n+1}q_n+q_{n-1}} = \frac{i\alpha_{n+2}+1}{\alpha_{n+2}q_{n+1}+q_n} > \frac{1}{q_{n+1}-iq_n}.$$

Since

$$\alpha > a+\frac{1}{a}-1 = \frac{p_1}{q_1}-1 \geq \frac{p_n}{q_n}-1 = \frac{q_{n+1}}{q_n}-1$$

for $n \geq 0$, we have

$$|(q_{n+1}-q_n)\alpha-(p_{n+1}-p_n)| = \frac{\alpha+1}{\alpha q_{n+1}+q_n} < \frac{1}{q_{n+1}-q_n},$$

yielding

$$\left|\alpha - \frac{p_{n+1}-p_n}{q_{n+1}-q_n}\right| < \frac{1}{(q_{n+1}-q_n)^2}.$$

Since

$$\alpha < a+1 = \frac{p_0}{q_0}+1 \leq \frac{p_n}{q_n}+1 = \frac{q_{n+1}}{q_n}+1$$

for $n \geq 1$, we have

$$\|(q_{n+1}-(a_{n+1}-1)q_n)\alpha\| = |(q_n+q_{n-1})\alpha-(p_n+p_{n-1})| = \frac{\alpha-1}{\alpha q_n+q_{n-1}} < \frac{1}{q_n+q_{n-1}},$$

yielding

$$\left|\alpha - \frac{p_n+p_{n-1}}{q_n+q_{n-1}}\right| < \frac{1}{(q_n+q_{n-1})^2}.$$

For $n=0$,

$$\left|\alpha - \frac{p_0+1}{q_0}\right| = |\alpha-(a+1)| < 1 = \frac{1}{q_0^2}.$$

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