THE r-SUBCOMPLETE PARTITIONS

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1. INTRODUCTION

MacMahon [1], pp. 217-223, studied special kinds of partitions of a positive integer, which he called perfect partitions and subperfect partitions. He defined a perfect partition of a number as "a partition which contains one and only one partition of every lesser number" and a subperfect partition as "a partition which contains one and only one partition of every lesser number if it is permissible to regard the several parts as affected with either the positive or negative sign". For instance, (1 1 3) is a perfect partition of 5 because we can uniquely express each of the numbers 1 through 5 by using the parts of two 1's and a 3. Thus (1), (1 1), (3), and (1 3) are the partitions referred to. The partition (1 3) is a subperfect partition of 4 because 1 is represented by the part 1, 2 by $-1+3$, and 3 3 by the part 3. In [1] MacMahon derived a recurrence relation for the number of such partitions using generating functions and found a nice relation between the number of perfect partitions and the number of ordered factorizations. See [4] for more information.

One way of generalizing MacMahon's idea is to eliminate the uniqueness condition, which was done by the second author [2]. He defines a complete partition of n to be a partition $\lambda=(\lambda_1\cdots\lambda_l)$ such that every number m with $1\leq m\leq n$ can be represented by the form of $m=\sum_{i=1}^{l}\alpha_i\lambda_i$, where $\alpha_i\in S=\{0,1\}$. He also studied the case of the set $S=\{0,1,\cdots,r\}$ in [3]. In this paper we shall study the r-subcomplete partitions which are complete partitions with the set $S=\{-r,\cdots,-1,0,1,\cdots,r\}$, where r is a positive integer.

2. THE r-SUBCOMPLETE PARTITIONS

Even if it is well-known, we start with a definition of partitions. Throughout this paper the number n represents a positive integer.

Definition 2.1: *A partition of n is a finite non-decreasing sequence $\lambda=(\lambda_1\cdots\lambda_l)$ such that $\sum_{i=1}^{l}\lambda_i=n$ and $\lambda_i>0$ for all $i=1,\cdots,l$. The λ_i are called the parts of the partition and the number l is called the length of the partition.*

We sometimes write $\lambda=(1^{m_1}2^{m_2}\cdots)$, which means there are exactly m_i parts equal to i in the partition λ. For example, we can write 7 partitions of 5 as (5), (1 4), (2 3), (1^2 3), (1 2^2), (1^3 2), and (1^5). The following two concepts are already mentioned, but we formally define them again to see how we can generalize them.

Definition 2.2: *A partition $\lambda=(\lambda_1^{m_1}\cdots\lambda_l^{m_l})$ of n is a perfect partition of n if every integer m with $1\leq m\leq n$ can be uniquely expressed as $m=\sum_{i=1}^{l}\alpha_i\lambda_i$, where $\alpha_i\in\{0,1,\cdots,m_i\}$ and repeated parts are regarded as indistinguishable.*

Definition 2.3: *A partition $\lambda=(\lambda_1^{m_1}\cdots\lambda_l^{m_l})$ of n is a subperfect partition if each integer m with $1\leq m\leq n$ can be uniquely represented as $\sum_{i=1}^{l}\alpha_i\lambda_i$, where $\alpha_i\in\{-m_i,\cdots,-1,0,1,\cdots,m_i\}$ and repeated parts are regarded as indistinguishable.*

Definition 2.4: *A partition of n* $\lambda = (\lambda_1 \cdots \lambda_l)$ *is a complete partition of n if each integer m with* $1 \leq m \leq n$ *can be represented as* $m = \sum_{i=1}^{l} \alpha_i \lambda_i$*, where* $\alpha_i \in \{0, 1\}$.

For $n = 6$, partitions $(1^6), (1^4\ 2), (1^3\ 3), (1^2\ 2^2)$ and $(1\ 2\ 3)$ are complete partitions. We refer to the paper [2] for more information on complete partitions. Now we are ready to define our main topic, the r-subcomplete partitions.

Definition 2.5: *A partition of n* $\lambda = (\lambda_1 \cdots \lambda_l)$ *is an r-subcomplete partition of n if each integer m with* $1 \leq m \leq rn$ *can be expressed as* $m = \sum_{i=1}^{l} \alpha_i \lambda_i$*, where* $\alpha_i \in \{-r, \cdots, -1, 0, 1, \cdots, r\}$*. Such as m is said to be r-representable.*

We also say a partition $\lambda = (\lambda_1 \cdots \lambda_l)$ is r-subcomplete if it is an r-subcomplete partition of the number $\lambda_1 + \cdots + \lambda_l$. We will write $\{0, \pm 1, \pm 2, \cdots, \pm r\}$ for the set $\{-r, \cdots, -1, 0, 1, \cdots, r\}$ and the letter r represents a positive integer throughout this paper. The r-subcomplete partitions with the set $\{0, 1, \cdots, r\}$ are called r-complete partitions. See [3] for more information.

Example 1: *The partition* $(1\ 4)$ *is a* 2*-subcomplete partition of* 5*. To see this we list* 2*-representations of numbers from* 1 *to* 10; $1 = 1$, $2 = 2 \cdot 1 + 0 \cdot 4, 3 = -1 \cdot 1 + 1 \cdot 4, 4 = 0 \cdot 1 + 1 \cdot 4, 5 = 1 \cdot 1 + 1 \cdot 4, 6 = 2 \cdot 1 + 1 \cdot 4, 7 = -1 \cdot 1 + 2 \cdot 4, 8 = 0 \cdot 1 + 2 \cdot 4, 9 = 1 \cdot 1 + 2 \cdot 4,$ *and* $10 = 2 \cdot 1 + 2 \cdot 4$.

It is easy to see that every integer m with $-rn \leq m \leq 0$ can also be expressed in the form $\sum_{i=1}^{l} \alpha_i \lambda_i$ with $\alpha_i \in \{0, \pm 1, \pm 2, \cdots, \pm r\}$ if $\lambda = (\lambda_1 \cdots \lambda_l)$ is an r-subcomplete partition of n. So one can say if $\lambda = (\lambda_1 \cdots \lambda_l)$ is an r-subcomplete partition of n then each number between $-rn$ and rn can be represented by the form. We will need this simple fact in the proof of Lemma 2.9 and Theorem 2.10. The following Lemma shows that every r-subcomplete partition should have 1 as the first part.

Lemma 2.6: *Let* $\lambda = (\lambda_1 \cdots \lambda_l)$ *be an r-subcomplete partition of n. Then* λ_1 *is 1.*

Proof: Suppose not. Then $\lambda_1 > 1$. Since λ is an r-subcomplete partition of n, the numbers 1 and $rn - 1$ are r-representable. Let $1 = \sum_{i=1}^{l} \alpha_i \lambda_i$, where $\alpha_i \in \{0, \pm 1, \pm 2, \cdots, \pm r\}$. Then there should be at least one $\alpha_j < 0$ for some j since $\lambda_i > 1$ for all i. Then $rn - 1 = r(\sum_{i=1}^{l} \lambda_i) - \sum_{i=1}^{l} \alpha_i \lambda_i = \sum_{i=1}^{l} (r - \alpha_i) \lambda_i$. Then $r - \alpha_j > r$, which means $rn - 1$ is not r-representable, which is a contradiction. □

Theorem 2.7: *Let* $\lambda = (\lambda_1 \cdots \lambda_l)$ *be an r-subcomplete partition of n. Then* $\lambda_i \leq 1 + 2r \sum_{j=1}^{i-1} \lambda_j$ *for each* $i = 2, \cdots, l$.

Proof: Suppose not. Then there exists at least one number k such that $\lambda_k > 1 + 2r \sum_{j=1}^{k-1} \lambda_j$, where $2 \leq k \leq l$. Thus,

$$rn > rn - \left(1 + 2r \sum_{j=1}^{k-1} \lambda_j\right) > rn - \lambda_k = r \sum_{j=1}^{l} \lambda_j - \lambda_k$$

$$= r \sum_{j=1}^{k-1} \lambda_j + (r-1)\lambda_k + r \sum_{j=k+1}^{l} \lambda_j.$$

Since the number $rn-\left(1+2r\sum_{j=1}^{k-1}\lambda_j\right)$ is r-representable, we can let $rn-\left(1+2r\sum_{j=1}^{k-1}\lambda_j\right)=\sum_{i=1}^{l}\alpha_i\lambda_i$, where $\alpha_i\in\{0,\pm1,\pm2,\cdots,\pm r\}$. Then $\alpha_k=\alpha_{k+1}=\cdots=\alpha_l=r$ because $rn-\left(1+2r\sum_{j=1}^{k-1}\lambda_j\right)>r\sum_{j=1}^{k-1}\lambda_j+(r-1)\lambda_k+r\sum_{j=k+1}^{l}\lambda_j$ and $\lambda_1\leq\cdots\leq\lambda_k\leq\lambda_{k+1}\leq\cdots\leq\lambda_l$. Thus,

$$rn-\left(1+2r\sum_{j=1}^{k-1}\lambda_j\right)=\sum_{j=1}^{k-1}\alpha_j\lambda_j+r\sum_{j=k}^{l}\lambda_j.$$

So

$$rn=1+\sum_{j=1}^{k-1}(2r+\alpha_j)\lambda_j+r\sum_{j=k}^{l}\lambda_j\geq rn+1,$$

which is a contradiction. □

Corollary 2.8: *Let $\lambda=(\lambda_1\cdots\lambda_l)$ be an r-subcomplete partition of n. Then $\lambda_i\leq(2r+1)^{i-1}$ for each $i=1,\cdots,l$.*

Proof: For $i=1$, the result is obvious. Assuming that $\lambda_i\leq(2r+1)^{i-1}$ for $i=1,\cdots,k$,

$$\lambda_{k+1}\leq 1+2r\sum_{j=1}^{k}\lambda_j\leq 1+2r\cdot\frac{(2r+1)^k-1}{(2r+1)-1}=(2r+1)^k. \quad \square$$

Lemma 2.9: *Let $\lambda=(\lambda_1\cdots\lambda_l)$ be an r-subcomplete partition of n. Then for $k=1,\cdots,l$ each partition $(\lambda_1\cdots\lambda_k)$ is r-subcomplete of the number $\lambda_1+\cdots+\lambda_k$.*

Proof: Clearly, (1) is an r-subcomplete partition of 1 for all r. Assume that $(\lambda_1\cdots\lambda_k)$ is an r-subcomplete partition of $\lambda_1+\cdots+\lambda_k$. We only need to show that for $w=1,\cdots,r$ each m such that $r(\lambda_1+\cdots+\lambda_k)+(w-1)\lambda_{k+1}<m\leq r(\lambda_1+\cdots+\lambda_k)+w\lambda_{k+1}$ is r-representable. Since $\lambda_{k+1}\leq 1+2r\sum_{j=1}^{k}\lambda_j$ from Theorem 2.7,

$$r(\lambda_1+\cdots+\lambda_k)-\lambda_{k+1}<m-w\lambda_{k+1}\leq r(\lambda_1+\cdots+\lambda_k)$$

$$-\left(1+r\sum_{j=1}^{k}\lambda_j\right)<m-w\lambda_{k+1}\leq r\sum_{j=1}^{k}\lambda_j$$

$$-r\sum_{j=1}^{k}\lambda_j\leq m-w\lambda_{k+1}\leq r\sum_{j=1}^{k}\lambda_j.$$

Thus by the inductive assumption, the number $m - w\lambda_{k+1}$ is r-representable by $\lambda_1, \cdots, \lambda_k$. So $m - w\lambda_{k+1} = \sum_{j=1}^{k} \alpha_j \lambda_j$, where $\alpha_j \in \{0, \pm 1, \pm 2, \cdots, \pm r\}$. Therefore, $m = \sum_{j=1}^{k} \alpha_j \lambda_j + w\lambda_{k+1}$. This completes the proof. $\square$

The converse of Theorem 2.7 is also true and it gives a criterion to determine r-subcomplete partitions.

Theorem 2.10: *Let $\lambda = (\lambda_1 \cdots \lambda_l)$ be a partition of n with $\lambda_1 = 1$ and $\lambda_i \leq 1 + 2r \sum_{j=1}^{i-1} \lambda_j$ for $i = 2, \cdots, l$. Then λ is an r-subcomplete partition of n.*

Proof: Obviously, (1) is an r-subcomplete partition of 1 for all r. Asusme that $(\lambda_1 \cdots \lambda_k)$ is r-subcomplete. Then by Lemma 2.9, every partition $(\lambda_1 \cdots \lambda_i)$ is r-subcomplete for $i = 2, \cdots, k$. We want to show that $(\lambda_1 \cdots \lambda_k \lambda_{k+1})$ is r-subcomplete. To do this we use similar steps to the proof of Lemma 2.9. Let m satisfy $r(\lambda_1 + \cdots + \lambda_k) + (w-1)\lambda_{k+1} < m \leq r(\lambda_1 + \cdots + \lambda_k) + w\lambda_{k+1}$, where $w = 1, 2, \cdots, r$. Now, since $r(\lambda_1 + \cdots + \lambda_k) - \lambda_{k+1} < m - w\lambda_{k+1} \leq r(\lambda_1 + \cdots + \lambda_k)$ and from the given condition $\lambda_{k+1} \leq 1 + 2r \sum_{j=1}^{k} \lambda_j$, we have $-r(\lambda_1 + \cdots + \lambda_k) \leq m - w\lambda_{k+1} \leq r(\lambda_1 + \cdots + \lambda_k)$. By the inductive assumption and Lemma 2.9 $m = w\lambda_{k+1} + \sum_{i=1}^{k} \alpha_i \lambda_i$, that is, $(\lambda_1 \cdots \lambda_k, \lambda_{k+1})$ is an r-subcomplete partition. Thus, the partition $(\lambda_1 \cdots \lambda_l)$ is an r-subcomplete partition. $\square$

Proposition 2.11: *Let $\lambda = (\lambda_1 \cdots \lambda_l)$ be an r-subcomplete partition of n. Then the minimum possible length l is $\lceil \log_{(2r+1)}(2rn+1) \rceil$, where $\lceil x \rceil$ is the least integer which is greater than or equal to x.*

Proof: By Corollary 2.8, $n = \sum_{j=1}^{l} \lambda_j \leq \sum_{j=1}^{l} (2r+1)^{j-1} = \frac{(2r+1)^l - 1}{2r}$. Therefore, $l \geq \lceil \log_{(2r+1)}(2rn+1) \rceil$. $\square$

Proposition 2.12: *Let $\lambda = (\lambda_1 \cdots \lambda_l)$ be an r-subcomplete partition of n. Then the largest possible part is $\lfloor \frac{2rn+1}{2r+1} \rfloor$, where $\lfloor x \rfloor$ is the largest integer which is less than or equal to x.*

Proof: Let $n = \lambda_1 + \cdots + \lambda_{l-1} + \lambda_l$. Then λ_l is the largest and $n - \lambda_l = \sum_{j=1}^{l-1} \lambda_j$. By Theorem 2.7, $\lambda_l \leq 1 + 2r \sum_{j=1}^{l-1} \lambda_j = 1 + 2r(n - \lambda_l)$. Thus $\lambda_l \leq \lfloor \frac{2rn+1}{2r+1} \rfloor$. $\square$

Now, we try to find two recurrence relations and a generating function for r-subcomplete partitions. Let $S_{r,k}(n)$ be the number of r-subcomplete partitions of n whose largest part is at most k. The set of such partitions can be partitioned into two subsets: one with the largest part at most $k-1$ and the other with the largest part exactly k. The latter type of partitions can be obtained by adding k as the last part of an r-subcomplete partitions of $n-k$ whose largest part is at most k. We know from Proposition 2.12 that the largest possible part k should satisfy $1 \leq k \leq \lfloor \frac{2rn+1}{2r+1} \rfloor$. If $k > \lfloor \frac{2rn+1}{2r+1} \rfloor$, $S_{r,k}(n)$ becomes actually $S_{r, \lfloor \frac{2rn+1}{2r+1} \rfloor}(n)$, which is the number of all r-subcomplete partitions of n. It is easy to see from the definition of $S_{r,k}(n)$ that $S_{r,k}(1) = 1$ for all k and $S_{r,1}(n) = 1$ for all n. Thus we obtain

Theorem 2.13: *Let $S_{r,k}(n)$ be the number of r-subcomplete partitions of n whose largest part is at most k. Then $S_{r,1}(n) = 1$ for all n and for $k \geq 2$*

$$S_{r,k}(n) = \begin{cases} S_{r,k-1}(n) + S_{r,k}(n-k) & \text{if } 1 \leq k \leq \lfloor \frac{2rn+1}{2r+1} \rfloor \\ S_{r, \lfloor \frac{2rn+1}{2r+1} \rfloor}(n) & \text{if } k > \lfloor \frac{2rn+1}{2r+1} \rfloor \end{cases}$$

with initial conditions $S_{r,1}(0) = 1$ *and* $S_{r,0}(n) = 0$ *for all* n.

Example 2:

$$\begin{aligned}
S_{2,6}(7) = S_{2,5}(7) &= S_{2,4}(7) + S_{2,5}(2) = (S_{2,3}(7) + S_{2,4}(3)) + S_{2,1}(2) \\
&= (S_{2,2}(7) + S_{2,3}(4)) + S_{2,2}(3) + 1 \\
&= \{(S_{2,1}(7) + S_{2,2}(5)) + (S_{2,2}(4) + S_{2,3}(1))\} + (S_{2,1}(3) + S_{2,2}(1)) + 1 \\
&= 1 + (S_{2,1}(5) + S_{2,2}(3)) + (S_{2,1}(4) + S_{2,2}(2)) + 1 + (1+1) + 1 \\
&= 1 + 1 + (S_{2,1}(3) + S_{2,2}(1)) + 1 + S_{2,1}(2) + 4 \\
&= 2 + (1 + S_{2,1}(1)) + 1 + 1 + 4 = 10
\end{aligned}$$

Now let us count the number of r-subcomplete partitions whose largest part is exactly k and find a generating function for this number. Let $E_{r,k}(n)$ be the number of r-subcomplete partitions of n whose largest part is exactly k. A recurrence relation for $E_{r,k}(n)$ can be obtained by the method we used in deriving $S_{r,k}(n)$ above, but we can use the number $S_{r,k}(n)$ itself as follows. From the recurrence relation for $S_{r,k}(n)$,

$$\begin{aligned}
E_{r,k}(n) &= S_{r,k}(n) - S_{r,k-1}(n) = S_{r,k}(n-k) \\
&= S_{r,k}(n-k) - S_{r,k-1}(n-k) + S_{r,k-1}(n-k) \\
&= E_{r,k}(n-k) + E_{r,k-1}(n-1).
\end{aligned}$$

It is easy to see that $E_{r,1}(n) = 1$ for all n. The numbers $E_{r,k}(n), E_{r,k-1}(n-1)$, and $E_{r,k}(n-k)$ count corresponding r-subcomplete partitions of $n, n-1$, and $n-k$, respectively. So they should satisfy the condition of Proposition 2.12. In other words, each of them must have $k \le \lfloor \frac{2rn+1}{2r+1} \rfloor, k-1 \le \lfloor \frac{2r(n-1)+1}{2r+1} \rfloor$, and $k \le \lfloor \frac{2r(n-k)+1}{2r+1} \rfloor$, respectively. Summarizing these, we obtain

Theorem 2.14: *Let* $E_{r,k}(n)$ *be the number of* r*-subcomplete partitions of a positive integer* n *whose largest part is exactly* k. *Then* $E_{r,1}(n) = 1$ *for all* n, *and for* $k \ge 2$

$$E_{r,k}(n) = \begin{cases} E_{r,k-1}(n-1) + E_{r,k}(n-k) & \text{if } n \ge 2k + \frac{k-1}{2r} \\ E_{r,k-1}(n-1) & \text{if } k + \frac{k-1}{2r} \le n < 2k + \frac{k-1}{2r} \\ 0 & \text{if } n < k + \frac{k-1}{2r} \end{cases}$$

with $E_{r,0}(0) = 1, E_{r,0}(n) = 0$ *for all* n *and* $E_{r,k}(n) = 0$ *for all* $n \le k$.

Example 3:

1. $E_{2,2}(5) = E_{2,1}(4) + E_{2,2}(3) = 1 + E_{2,1}(2) = 2.$
2. $E_{2,2}(6) = E_{2,1}(5) + E_{2,2}(4) = 1 + E_{2,1}(3) = 2.$
3. $E_{2,3}(9) = E_{2,2}(8) + E_{2,3}(6) = E_{2,1}(7) + E_{2,2}(6) + E_{2,2}(5) = 1 + 2 + 2 = 5.$
4. $E_{2,4}(6) = E_{2,3}(5) + E_{2,2}(4) = E_{2,1}(3) = 1.$
5. $E_{2,4}(5) = E_{2,3}(4) = E_{2,2}(3) = E_{2,1}(2) = 1.$
6. $E_{2,5}(11) = E_{2,4}(10) + E_{2,5}(6) = (E_{2,3}(9) + E_{2,4}(6)) + E_{2,4}(5) = 5 + 1 + 1 = 7.$

The following three tables show the first few values of r-subcomplete partitions with $r = 1, 2$ and 3. We denote $S_r(n)$ as the number of all r-subcomplete partitions of n.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2			1	1	2	2	3	3	4	4	5
3				1	1	2	3	4	5	7	8
4						1	2	3	4	6	9
5							1	2	3	4	6
6									2	3	4
7										2	3
$S_1(n)$	1	1	2	3	4	6	10	13	19	27	36

Table I $r = 1$

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2			1	1	2	2	3	3	4	4	5
3				1	1	2	3	4	5	7	8
4					1	1	2	3	5	6	9
5						1	1	2	3	5	7
6								1	2	3	5
7									1	2	3
8										1	2
9											1
$S_2(n)$	1	1	2	3	5	7	10	14	21	29	41

Table II $r = 2$

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2			1	1	2	2	3	3	4	4	5
3				1	1	2	3	4	5	7	8
4					1	1	2	3	5	6	9
5						1	1	2	3	5	7
6							1	1	2	3	5
7								1	1	2	3
8										1	2
9											1
$S_3(n)$	1	1	2	3	5	7	11	15	21	29	41

Table III $r = 3$

Now we are ready to find a generating function for $E_{r,k}(n)$. Based on Theorem 2.14 we obtain the following.

Theorem 2.15: *Let* $F_{r,k}(q) = \sum_{n=0}^{\infty} E_{r,k}(n)q^n$ *for* $k \geq 1$. *Then*

$$F_{r,k}(q) = \frac{q^k}{(q)_k}\left[1 - \sum_{i=0}^{s-1} q^i (q)_{2ri+1} E_{r,2ri+1}((2r+1)i+1)\right], \tag{2.1}$$

where $s = \lceil \frac{k-1}{2r} \rceil$, $F_{r,0}(q) = 1$, *and* $(q)_k = (1-q^k)(1-q^{k-1})\cdots(1-q)$.

Proof: By Theorem 2.14,

$$\begin{aligned}
F_{r,k}(q) &= \sum_{n=k+s}^{\infty} E_{r,k}(n)q^n \\
&= \sum_{n=k+s}^{2k+s-1} E_{r,k-1}(n-1)q^n + \sum_{n=2k+s}^{\infty} [E_{r,k-1}(n-1) + E_{r,k}(n-k)]q^n \\
&= \sum_{n=k+s}^{\infty} E_{r,k-1}(n-1)q^n + \sum_{n=2k+s}^{\infty} E_{r,k}(n-k)q^n \\
&= q \sum_{n=k+s-1}^{\infty} E_{r,k-1}(n)q^n + q^k \sum_{n=k+s}^{\infty} E_{r,k}(n)q^n.
\end{aligned} \tag{2.2}$$

Since $s = \lceil \frac{k-1}{2r} \rceil$, its value depends on k and r. Thus from Proposition 2.12, $F_{r,k-1}(q)$ becomes $F_{r,k-1}(q) = \sum_{n=k+s-1}^{\infty} E_{r,k-1}(n)q^n$ or $F_{r,k-1}(q) = \sum_{n=k+s-2}^{\infty} E_{r,k-1}(n)q^n$. Let $2rp+3 \leq k \leq 2r(p+1)+1$ for some $p = 0, 1, 2, \cdots$. Then $2rp+2 \leq k-1 \leq 2r(p+1)$ and $s = p+1$. Consider $E_{r,k-1}(k+s-2)$. This is the number of r-subcomplete partitions of $k+s-2 = k-1+p$ whose largest part is exactly $k-1$. So any number between 1 and $r(k-1+p)$ should be r-representable. But with $k-1 = 2rp+t (2 \leq t \leq 2r)$ fixed as the largest part, the number $rp+1$ can not be r-representable. Thus, $E_{r,k-1}(k+s-2) = 0$ for $2rp+2 \leq k-1 \leq 2r(p+1)$. Thus, we obtain

$$F_{r,k-1}(q) = \begin{cases} \sum_{n=k+s-1}^{\infty} E_{r,k-1}(n)q^n & \text{if } k \not\equiv 2 (\text{mod } 2r) \\ \sum_{n=k+s-2}^{\infty} E_{r,k-1}(n)q^n & \text{if } k \equiv 2 (\text{mod } 2r). \end{cases}$$

For $k \not\equiv 2(\text{mod } 2r)$ equation (2.2) becomes

$$F_{r,k}(q) = q \sum_{n=k+s-1}^{\infty} E_{r,k-1}(n)q^n + q^k \sum_{n=k+s}^{\infty} E_{r,k}(n)q^n = qF_{r,k-1}(q) + q^k F_{r,k}(q).$$

Thus,

$$F_{r,k}(q) = \frac{q}{1-q^k} F_{r,k-1}(q). \tag{2.3}$$

For $k \equiv 2 (\text{mod } 2r)$ equation (2.2) becomes

$$\begin{aligned} F_{r,k}(q) &= q \sum_{n=k+s-1}^{\infty} E_{r,k-1}(n) q^n + q^k \sum_{n=k+s}^{\infty} E_{r,k}(n) q^n \\ &= q\{F_{r,k-1}(q) - E_{k-1}(k+s-2) q^{k+s-2}\} + q^k F_{r,k}(q). \end{aligned}$$

Thus we have

$$F_{r,k}(q) = \frac{q}{1-q^k} F_{r,k-1}(q) - \frac{q^{k+s-1} E_{r,k-1}(k+s-2)}{1-q^k}. \tag{2.4}$$

Now, let $k = 2rp + 2 + t$ for some non-negative integer p with $1 \le t \le 2r-1$. Then $k \not\equiv 2(\text{mod } 2r)$, so we can iterate equation (2.3) t times to get

$$F_{r,k}(q) = \frac{q^t}{(1-q^k)(1-q^{k-1})\cdots(1-q^{k-t+1})} F_{r,k-t}(q). \tag{2.5}$$

Because $k - t = 2rp + 2 \equiv 2(\text{mod } 2r)$, we have to use identity (2.4) to compute $F_{r,k-t}(q)$ which is equal to $F_{r,2r(s-1)+2}(q)$ since $\lceil \frac{2rp+2}{2r} \rceil = p + 1 = s$. We have

$$\begin{aligned} F_{r,2r(s-1)+2}(q) = &\frac{q}{1-q^{2r(s-1)+2}} F_{r,2r(s-1)+1}(q) - \\ &\frac{q^{2r(s-1)+s+1} E_{r,2r(s-1)+1}(2r(s-1)+s)}{1-q^{2r(s-1)+2}}. \end{aligned} \tag{2.6}$$

Thus by applying (2.6) to (2.5),

$$\begin{aligned} F_{r,k}(q) = &\frac{q^{k-2r(s-1)-1}}{(1-q^k)\cdots(1-q^{2r(s-1)+2})} F_{r,2r(s-1)+1}(q) - \\ &\frac{q^{k+s-1} E_{r,2r(s-1)+1}(2r(s-1)+s)}{(1-q^k)\cdots(1-q^{2r(s-1)+2})}. \end{aligned} \tag{2.7}$$

Now $2r(s-1)+1 = 2rp+1 \not\equiv 2 (\text{mod } 2r)$, so by equation (2.3)

$$F_{r,2r(s-1)+1}(q) = \frac{q}{1-q^{2r(s-1)+1}} F_{r,2r(s-1)}(q).$$

Again this can be iterated $2r-1$ times, which gives us

$$F_{r,2r(s-1)+1}(q) = \frac{q^{2r-1}}{(1-q^{2r(s-1)+1})\cdots(1-q^{2r(s-2)+3})} F_{r,2r(s-2)+2}(q). \tag{2.8}$$

By applying (2.8) to (2.7),

$$F_{r,k}(q) = \frac{q^{k-2r(s-2)-2}}{(1-q^k)\cdots(1-q^{2r(s-2)+3})} F_{r,2r(s-2)+2}(q) - \frac{q^{k+s-1} E_{r,2r(s-1)+1}(2r(s-1)+s)}{(1-q^k)\cdots(1-q^{2r(s-1)+2})}. \tag{2.9}$$

The number $2r(s-2)+2 = 2r(p-1)+2 \equiv 2 (\text{mod } 2r)$. So by (2.4) and with $\lceil \frac{2r(s-2)+1}{2r} \rceil = s-1$,

$$F_{r,2r(s-2)+2}(q) = \frac{q}{(1-q^{2r(s-2)+2})} F_{r,2r(s-2)+1}(q) - \frac{q^{2r(s-2)+s} E_{r,2r(s-2)+1}(2r(s-2)+s-1)}{1-q^{2r(s-2)+2}}. \tag{2.10}$$

Thus,

$$F_{r,k}(q) = \frac{q^{k-2r(s-2)-1}}{(1-q^k)\cdots(1-q^{2r(s-2)+2})} F_{r,2r(s-2)+1}(q) - \frac{q^{k+s-2} E_{r,2r(s-2)+1}(2r(s-2)+s-1)}{(1-q^k)\cdots(1-q^{2r(s-2)+2})} - \frac{q^{k+s-1} E_{r,2r(s-1)+1}(2r(s-1)+s)}{(1-q^k)\cdots(1-q^{2r(s-1)+2})}. \tag{2.11}$$

By continuing iteration on $F_{r,2r(s-2)+1}(q)$, we finally obtain the following.

$$F_{r,k}(q) = \frac{q^k}{(q)_k} - \left[\frac{q^k E_{r,1}(1)}{(1-q^k)\cdots(1-q^2)} + \frac{q^{k+1}E_{r,2r+1}(2r+2)}{(1-q^k)\cdots(1-q^{2r+2})} + \cdots \right.$$
$$+ \frac{q^{k+s-2}E_{r,2r(s-2)+1}(2r(s-2)+s-1)}{(1-q^k)\cdots(1-q^{2r(s-2)+2})} +$$
$$\left. \frac{q^{k+s-1}E_{r,2r(s-1)+1}(2r(s-1)+s)}{(1-q^k)\cdots(1-q^{2r(s-1)+2})}\right]. \tag{2.12}$$

One can easily derive our formula (2.1) from this result. □

Example 4: *The following are generating functions for $k = 2, 4, 5, 6$ and $r = 2$.*

$$F_{2,2}(q) = \frac{q^2}{(q)_2} - \frac{q^2 E_{2,1}(1)}{1-q^2} = \frac{q^2}{(q)_2} - \frac{q^2}{1-q^2}.$$

$$F_{2,4}(q) = \frac{q}{(1-q^4)}F_{2,3} = \frac{q^2}{(1-q^4)(1-q^3)}F_{2,2}(q) = \frac{q^5}{(q)_4}.$$

$$F_{2,5}(q) = \frac{q}{(q)_5}F_{2,4}(q) = \frac{q^6}{(q)_5}.$$

$$F_{2,6}(q) = \frac{q^6}{(q)_6} - \left[\frac{q^7 E_{2,5}(6)}{1-q^6} + \frac{q^6 E_{2,1}(1)}{(1-q^6)(1-q^5)(1-q^4)(1-q^3)(1-q^2)}\right]$$
$$= \frac{q^6}{(q)_6} - \left[\frac{q^7}{1-q^6} + \frac{q^6}{(1-q^6)(1-q^5)(1-q^4)(1-q^3)(1-q^2)}\right].$$

Example 5: *By expanding the above, we obtain the following generating functions whose coefficients are expected from* Table II.

$$F_{2,2}(q) = q^3 + q^4 + 2q^5 + 2q^6 + 3q^7 + 3q^8 + 4q^9 + 4q^{10} + 5q^{11} + 5q^{12} + 6q^{13} \cdots,$$
$$F_{2,4}(q) = q^5 + q^6 + 2q^7 + 3q^8 + 5q^9 + 6q^{10} + 9q^{11} + 11q^{12} + 15q^{13} + \cdots,$$
$$F_{2,5}(q) = q^6 + q^7 + 2q^8 + 3q^9 + 5q^{10} + 7q^{11} + 10q^{12} + 13q^{13} + \cdots,$$
$$F_{2,6}(q) = q^8 + 2q^9 + 3q^{10} + 5q^{11} + 7q^{12} + 10q^{13} + \cdots.$$

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