CERTAIN LUCAS-LIKE SEQUENCES AND THEIR GENERATION BY PARTITIONS OF NUMBERS

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1. INTRODUCTION

An interesting paper by S. L. Basin in the April, 1964, issue of this journal [1] develops the k^{th} Lucas number $L_k = S_k$ where S_k is the sum of the k^{th} powers of the roots of

$$(1) \qquad f(x) = a_0x^2 + a_1x + a_2 \quad ,$$

in which $a_0 = 1$, $a_1 = a_2 = -1$. Although Basin's S_k originated from a demonstration of a property of Waring's formula, it is obvious, as Basin implies, that the same results could be obtained using Newton's formulas for S_k in terms of elementary symmetric functions.

In a previous paper [2], the author tabulated S_k from Newton's formulas for

$$(2) \qquad f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \quad .$$

The values of S_k for $k = 1(1)11$ applicable for $1 \leq n \leq 11$ are reproduced as Table 1* of this paper.

It is proposed to examine the special case of (2),

$$(3) \qquad f(x) = x^n - x^{n-1} - x^{n-2} - \cdots - 1 \quad ,$$

for $n \geq 2$ and to use Table 1 as a guide in extending the true Lucas sequence found from (1) to Lucas-like sequences. Also, a method by which partitions of numbers can generate terms of the Lucas-like sequences is presented.

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*This table is reproduced with all rights reserved. Reprinted by permission from the American Mathematical Society from Mathematics of Computation, Vol. 12, No. 63, pp. 194-198. Actually, it is S_n which is tabulated in [2] but is presented herein as S_k to be consistent with this paper.

Table 1

S_k for k = 1(1)11

$$
\begin{aligned}
S_1 &= -\,a_1/a_0, \\
S_2 &= a_1^2/a_0^2 - 2a_2/a_0, \\
S_3 &= -\,a_1^3/a_0^3 + 3a_1a_2/a_0^2 - 3a_3/a_0, \\
S_4 &= a_1^4/a_0^4 - 4a_1^2a_2/a_0^3 + (4a_1a_3 + 2a_2^2)/a_0^2 - 4a_4/a_0, \\
S_5 &= -\,a_1^5/a_0^5 + 5a_1^3a_2/a_0^4 - (5a_1^2a_3 + 5a_1a_2^2)/a_0^3 + (5a_1a_4 + 5a_2a_3)/a_0^2 - 5a_5/a_0, \\
S_6 &= a_1^6/a_0^6 - 6a_1^4a_2/a_0^5 + (6a_1^3a_3 + 9a_1^2a_2^2)/a_0^4 - (6a_1^2a_4 + 12a_1a_2a_3 + 2a_2^3)/a_0^3 \\
&\quad + (6a_1a_5 + 6a_2a_4 + 3a_3^2)/a_0^2 - 6a_6/a_0, \\
S_7 &= -\,a_1^7/a_0^7 + 7a_1^5a_2/a_0^6 - (7a_1^4a_3 + 14a_1^3a_2^2)/a_0^5 \\
&\quad + (7a_1^3a_4 + 21a_1^2a_2a_3 + 7a_1a_2^3)/a_0^4 - (7a^2_1a_5 + 14a_1a_2a_4 + 7a^2_2a_3 + 7a_1a^2_3)/a_0^3 \\
&\quad + (7a_1a_6 + 7a_2a_5 + 7a_3a_4)/a_0^2 - 7a_7/a_0, \\
S_8 &= a_1^8/a_0^8 - 8a_1^6a_2/a_0^7 + (8a_1^5a_3 + 20a_1^4a_2^2)/a_0^6 \\
&\quad - (8a_1^4a_4 + 32a_1^3a_2a_3 + 16a_1^2a^3_2)/a_0^5 \\
&\quad + (8a_1^3a_5 + 24a_1^2a_2a_4 + 12a_1^2a_3^2 + 24a_1a_2^2a_3 + 2a_2^4)/a_0^4 \\
&\quad - (8a_1^2a_6 + 16a_1a_2a_5 + 16a_1a_3a_4 + 8a_2^2a_4 + 8a_2a_3^2)/a_0^3 \\
&\quad + (8a_1a_7 + 8a_2a_6 + 8a_3a_5 + 4a_4^2)/a_0^2 - 8a_8/a_0, \\
S_9 &= -\,a_1^9/a_0^9 + 9a_1^7a_2/a_0^8 - (9a_1^6a_3 + 27a_1^5a_2^2)/a_0^7 \\
&\quad + (9a_1^5a_4 + 45a_1^4a_2a_3 + 30a_1^3a_2^3)/a_0^6 \\
&\quad - (9a_1^4a_5 + 36a_1^3a_2a_4 + 18a_1^3a_3^2 + 54a_1^2a_2^2a_3 + 9a_1a_2^4)/a_0^5 \\
&\quad + (9a_1^3a_6 + 27a_1^2a_2a_5 + 27a_1^2a_3a_4 + 27a_1a_2^2a_4 + 27a_1a_2a_3^2 + 9a_2^3a_3)/a_0^4 \\
&\quad - (9a_1^2a_7 + 18a_1a_2a_6 + 18a_1a_3a_5 + 9a_1a_4^2 + 9a_2^2a_5 + 18a_2a_3a_4 + 3a_3^3)/a_0^3 \\
&\quad + (9a_1a_8 + 9a_2a_7 + 9a_3a_6 + 9a_4a_5)/a_0^2 - 9a_9/a_0, \\
S_{10} &= a_1^{10}/a_0^{10} - 10a_1^8a_2/a_0^9 + (10a_1^7a_3 + 35a_1^6a_2^2)/a_0^8 \\
&\quad - (10a_1^6a_4 + 60a_1^5a_2a_3 + 50a_1^4a_2^3)/a_0^7 + (10a_1^5a_5 + 50a_1^4a_2a_4 \\
&\quad + 25a_1^4a_3^2 + 100a_1^3a_2^2a_3 + 25a_1^2a_2^4)/a_0^6 - (10a_1^4a_6 + 40a_1^3a_2a_5 \\
&\quad + 40a_1^3a_3a_4 + 60a_1^2a_2^2a_4 + 60a_1^2a_3^2a_2 + 40a_2^3a_3a_1 + 2a_2^5)/a_0^5 \\
&\quad + (10a_1^3a_7 + 30a_1^2a_2a_6 + 30a_1^2a_3a_5 + 15a_1^2a_4^2 + 30a_1a_2^2a_5 \\
&\quad + 60a_1a_2a_3a_4 + 10a_2^3a_4 + 15a_2^2a_3^2 + 10a_1a_3^3)/a_0^4 \\
&\quad - (10a_1^2a_8 + 20a_1a_2a_7 + 20a_1a_3a_6 + 20a_1a_4a_5 + 20a_2a_3a_5 \\
&\quad + 10a_2a_4^2 + 10a_2^2a_6 + 10a_3^2a_4)/a_0^3 + (10a_1a_9 + 10a_2a_8 \\
&\quad + 10a_3a_7 + 10a_4a_6 + 5a_5^2)/a_0^2 - 10a_{10}/a_0, \\
S_{11} &= -\,a_1^{11}/a_0^{11} + 11a_1^9a_2/a_0^{10} - (11a_1^8a_3 + 44a_1^7a_2^2)/a_0^9 \\
&\quad + (11a_1^7a_4 + 77a_1^6a_2a_3 + 77a_1^5a_2^3)/a_0^8 - (11a_1^6a_5 + 66a_1^5a_2a_4 \\
&\quad + 33a_1^5a^2_3 + 165a_1^4a_2^2a_3 + 55a_1^3a_2^4)/a_0^7 + (11a_1^5a_6 + 55a_1^4a_2a_5 \\
&\quad + 55a_1^4a_3a_4 + 110a_1^2a_2^3a_3 + 110a_1^3a_2^2a_4 + 110a_1^3a_2a_3^2 + 11a_1a_2^5)/a_0^6 \\
&\quad - (11a_1^4a_7 + 44a_1^3a_2a_6 + 44a_1^3a_3a_5 + 22a_1^3a_4^2 + 66a_1^2a_2^2a_5 \\
&\quad + 132a_1^2a_2a_3a_4 + 44a_1a_2^3a_4 + 66a_1a_2^2a_3^2 + 11a_2^4a_3 + 22a_1^2a_3^3)/a_0^5 \\
&\quad + (11a_1^3a_8 + 33a_1^2a_2a_7 + 33a_1^2a_3a_6 + 33a_1^2a_4a_5 + 33a_1a_2^2a_6 \\
&\quad + 66a_1a_2a_3a_5 + 33a_1a_2a_4^2 + 33a_1a_3^2a_4 + 11a_2^3a_5 + 33a_2^2a_3a_4 + 11a_3^3a_2)/a_0^4 \\
&\quad - (11a_1^2a_9 + 22a_1a_2a_8 + 22a_1a_3a_7 + 22a_1a_4a_6 + 11a_1a_5^2 + 11a_2^2a_7 \\
&\quad + 22a_2a_3a_6 + 22a_2a_4a_5 + 11a_3a_4^2 + 11a_3^2a_5)/a_0^3 + (11a_1a_{10} + 11a_2a_9 \\
&\quad + 11a_3a_8 + 11a_4a_7 + 11a_5a_6)/a_0^2 - 11a_{11}/a_0.
\end{aligned}
$$

The liberty of calling the sequence "Lucas-like" appears justified since (1) (as used by Basin) is a special case of (3) and, moreover, the sequences do indeed share characteristics with the true Lucas sequences.

2. OBSERVED BEHAVIOR

To identify terms of a sequence and at the same time to retain a Lucas flavor, the terminology $L_k^{(n)}$ is used to specify the k^{th} term of a Lucas-like sequence obtained from (3) for a given $n \geq 2$. For convenience, $S_k^{(n)} = L_k^{(n)}$. It is noted that $L_k^{(2)}$ is the true k^{th} Lucas number, L_k. For any given $k \leq 11$ and $2 \leq n \leq 11$, it is a simple matter to enter Table 1, reject all coefficients of a terms having subscripts greater than n and add the numerical coefficients of the remaining a terms to obtain a k^{th} Lucas-like number. The choice of signs in (3) automatically makes the signs of the numerical coefficients positive. For example:

$$L_4^{(2)} = 1 + 4 + (0 + 2) + 0 = 7 \; . \tag{4}$$

The first seven terms of several Lucas-like sequences obtained in this manner are recorded in Table 2. For later use, a zig-zag line divides the table into two parts. For $n = 2$, it is seen that the difference between the first <u>two</u> terms (those above the zig-zag line) is 2 (i.e., 2^{n-1} for $n = 2$), and that the sum of <u>two</u> consecutive terms determines the next term. For $n = 3$, the difference between the two first terms is 2^{n-2}, and the difference between the second and third terms is 2^{n-1}. There are <u>three</u> terms above the zig-zag line. For $n = 3$, the sum of <u>three</u> consecutive terms determines the next term.

Table 2

VALUES OF $L_k^{(n)}$

k \ n	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	3	3	3	3	3
3	1	4	7	7	7	7
4	1	7	11	15	15	15
5	1	11	21	26	31	31
6	1	18	39	51	57	63
7	1	29	71	99	113	120

The obvious pattern is repeated for $n = 4, 5$, etc.

One immediate conclusion is that each Lucas-like sequence is, in reality, the blend of two sequences. The first sequence is $1, 3, 7, \cdots, 2^n - 1$ having n terms and governed within its range by the recursion formula

$$L_{k+1}^{(n)} = L_k^{(n)} + 2^k, \qquad (L_1^{(n)} = 1) \ . \tag{5}$$

The second sequence starts with the sum of the n terms of the first sequence. The first term is

$$L_{n+1}^{(n)} = 2^{n+1} - n - 2 \ . \tag{6}$$

Succeeding terms are

$$L_{n+2}^{(n)} = L_{n+1}^{(n)} + (2^{n+1} - n - 2) - 1 \ , \tag{7}$$

$$L_{n+3}^{(n)} = L_{n+2}^{(n)} + (2^{n+1} - n - 2) - (1 + 3) \ , \tag{8}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$L_{2n+1}^{(n)} = L_{2n-1}^{(n)} + \cdots + L_{n+1}^{(n)} \ . \tag{9}$$

In general, the second sequence follows the recursion formula

$$L_k^{(n)} = L_{k-1}^{(n)} + L_{k-2}^{(n)} + \cdots + L_{k-n}^{(n)} \quad (k \geq n + 1) \ . \tag{10}$$

It is interesting to note from (7), (8), and (9) that at least one term of the first sequence appears directly in the summation for $L_k^{(n)}$ for $n \leq k \leq 2n$. After $k > 2n$, the influence of the first sequence is reduced.

3. PARTITION CALCULATION OF SEQUENCE TERMS

Several methods are available for finding a particular $L_k^{(n)}$. One method is the direct use of recursion formulas. Another is to solve the n^{th} order difference equation for the second sequence subject to the n conditions (or their equivalents) imposed by the first sequence. A third method, discussed herein, is to assume that desired partitions of n are available and to use them as a combinatorial means of finding the $L_k^{(n)}$.

In Chrystal's [3] notation, $P(k|p|\leq q)$ is the number of p-part partitions of k, no member of which exceeds q. If the original value of q exceeds $k + 1 - p$, it can be replaced by $q = k + 1 - p$ since there are the same number of partitions for $q = k + 1 - p$ as for $q \leq k + 1 - p$. However, for $q \leq k + 1 - p$ it is obvious that less than $P(k|p|\leq k + 1 - p)$ partitions exist. Suppose, now, that desired partitions can be called up at will and are available from this point on. The actual set of such partitions which have the same limitations as the enumeration counterpart is given the terminology $PV(k|p|\leq q)$.

If any S_k of Table 1 is stripped of all terms except subscripts and superscripts (exponents) of the numerator a's, there remains the conventional representation of all the partitions of k. The partition representation for $k = 6$ is exemplified in Table 3. It is seen that, in general, p ranges from k to 1. The quantity $k \cdot (p - 1)!$ divided by the product of the factorials of the exponents of a particular combination yields (neglecting sign) the numerical part of the contribution of that combination to S_k. To illustrate, if $k = 6$, $p = 3$, the numerical coefficient associated with the partition $2^3 = 2, 2, 2$, is $(6 \times 2!)/3! = 2$. This well-known result employs much the same reasoning as finding a coefficient of a multinomial expansion. The numerical coefficients for $k = 6$, $n = 6$, and the total $63 = L_6^{(6)}$ are given in Table 3. Thus, once the exponents are found from the available partitions, $L_k^{(n)}$ follows.

Table 3

Partition Representation		$PV(k\|p\|\leq q)$	Numerical Coefficient
1^6	1, 1, 1, 1, 1, 1	$PV(k\|p\|\leq q)$	$(6 \times 5!)/6! = 1$
$1^4, 2$	1, 1, 1, 1, 2	$PV(6\|5\|\leq 2)$	$(6 \times 4!)/4! = 6$
$1^3, 3$	1, 1, 1, 3	$PV(6\|4\|\leq 3)$	$(6 \times 3!)/3! = 6$
$1^2, 2^2$	1, 1, 2, 2		$(6 \times 3!)/(2! \times 2!) = 9$
$1^2 4$	1, 1, 4	$PV(6\|3\|\leq 4)$	$(6 \times 2!)/2! = 6$
1, 2, 3	1, 2, 3		$(6 \times 2!)/1 = 12$
2^3	2, 2, 2		$(6 \times 2!)/3! = 2$
1, 5	1, 5	$PV(6\|2\|\leq 5)$	$(6 \times 1!)/1 = 6$
24	2, 4		$(6 \times 1!)/1 = 6$
3^2	3, 3		$(6 \times 1!)/2! = 3$
6	6	$PV(6\|1\|\leq 6)$	$(6 \times 0!)/1 = 6$
			Total = 63

As long as $k \le n$, the sum of numerical coefficients obtained from the $PV(k\,p \le k + 1 - p)$'s is the desired $L_k^{(n)}$. When $k \ge n$, the a terms with subscripts greater than n are zero. Since the corresponding products with numerical coefficients are zero, these numerical coefficients are not used. The elimination of these numerical coefficients is accomplished by limiting q to $1 \le q \le n$ and using only those partitions which result. Table 4 gives an example of this situation $k = 6$, $n = 2$.

Table 4

Partition Representation		PV(k|p|q)	Numerical Coefficient
1^6	1, 1, 1, 1, 1, 1	PV(6|6|1)	1
$1^4, 2$	1, 1, 1, 1, 2	PV(6|5|2)	6
$1^2, 2^2$	1, 1, 2, 2	PV(6|4|2)	9
2^3	2, 2, 2	PV(6|3|2)	2
None		PV(6|2|2)	0
None		PV(6|1|2)	0
			Total = 18

The above methods have been successfully applied to digital computation of electrical network problems [4] in which the a coefficients had values other than ±1 and in which it was necessary to consider the signs of the resultant numerical coefficients.

REFERENCES

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