THE EXISTENCE OF PERFECT 3-SEQUENCES

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For s and n positive integers, a sequence $a_1, a_2, \cdots, a_{sn}$ of length sn is called a perfect s-sequence for the integer n if (a) each of the integers $1, 2, \cdots, n$ occurs exactly s times in the sequence and (b) between any two consecutive occurrences of the integer i there are exactly i entries. Thus 4 1 3 1 2 4 3 2 is a perfect 2-sequence for $n = 4$. The problem of determining all n having a perfect s-sequence is posed in [1] for $s = 2$ and in [4] for $s > 2$.

It is shown in [3] that a perfect 2-sequence exists for an integer n if and only if $n = 3$ or $4 \pmod 4$, and furthermore, an explicit 2-sequence is presented for each such n.

The question of the existence of a perfect s-sequence for any n with $s > 2$ is then raised in [4] and [5]. The problem is partially answered in [5] by providing necessary conditions on n in the case where s is either a multiple of 2 or 3. In the particular case $s = 3$, a necessary condition that there exist a perfect 3-sequence for n is $n \equiv 1, 0,$ or $1 \pmod 9$.

The following examples lead one to believe that for $s = 3$, the above conditions are almost sufficient. Namely, we exhibit perfect 3-sequences for $n = 9, 10, 17, 18,$ and 19.

The case $n = 9$:

1 9 1 6 1 8 2 5 7 2 6 9 2 5 8 4 7 6 3 5 4 9 3 8 7 4 3

The case $n = 10$ (with 10 denoted by ϕ):

1 ϕ 1 6 1 7 9 3 5 8 6 3 ϕ 7 5 3 9 6 8 4 5 7 2 ϕ 4 2 9 8 2 4

The case $n = 17$:

17	15	3	16	9	10	3	1	12	1	3	1	13	14	9	6	10
15	17	5	16	12	6	11	9	5	13	10	14	6	7	5	8	15
12	11	17	16	7	4	13	8	2	14	4	2	7	11	2	4	8

The case $n = 18$:

18	16	5	17	11	4	2	9	5	2	4	14	2	15	5	4	11	9
16	18	12	17	13	6	7	8	14	9	11	15	6	10	7	12	8	16
13	6	18	17	7	14	10	8	3	15	12	1	3	1	13	1	3	10

The case $n = 19$:

19	17	13	18	4	11	8	2	16	4	2	9	15	2	4	8	13	11	14
17	19	9	18	12	8	16	5	7	15	11	13	9	5	14	10	7	12	17
5	6	19	18	16	7	15	10	6	3	14	12	1	3	1	6	1	3	10

From the above examples, one has

Conjecture. For $n > 8$, a necessary and sufficient condition that there exist a perfect 3-sequence for n is $n \equiv -1, 0, 1 \pmod 9$.

The necessary condition stated in the above conjecture is proved in [5]. Actually, the results of [5] are a special case of:

Theorem 1. Let $s = pt$ where p is a prime. A necessary condition that a perfect s-sequence exist for an integer n is

$$n \equiv -1, 0, 1, 2, \cdots, \text{ or } p - 2 \pmod{p^2} .$$

Proof. Suppose a perfect s-sequence $a_1, \cdots, a_{sn}$ exists. Then for an integer i occupying positions $c_1, c_2, \cdots, c_s$, we have

$$c_j = c_1 + (j - 1)(i + 1) \quad (j = 1, \cdots, s) .$$

If $i \not\equiv -1 \pmod p$, the positions c_j range over the residue classes mod p in a manner such that each residue class has an equal number t of occurrences.

On the other hand, for a fixed i such that $i \equiv -1 \pmod p$ the positions c_j are all congruent to each other mod p. Letting r be a residue of p, $0 \le r \le p - 1$, we define $N(r)$ as the number of integers $i \equiv -1 \pmod p$ such that the common residue of $c_1, \cdots, c_s$ is r.

We now let $b(n, p)$ denote the number of integers i such that $1 \le i \le n$ and $i \equiv -1 \pmod p$. Then, observing that the total number of positions in the sequence $a_1, \cdots, a_{sn}$ congruent to $r \pmod p$ must be nt, it follows (by counting the number of such positions filled by integers i in the range $1 \le i \le n$) that

$$t \cdot b(n,p) + sN(r) = nt .$$

Thus, all $N(r)$ have the common value N expressed by

$$pN = n - b(n,p) = \left[\frac{n+1}{p}\right]$$

where [] is the greatest integer function. Representing n by $n = kp + q$ with $-1 \le q \le p - 2$ it follows that $pN = k$ and $n = p^2N + q$, whence n is out in the assumed range of values.

The fact that theorem 1 is in some sense strong for $s = 3$ does not completely reflect what conditions are required on n for $s > 3$. In particular, if a power (greater than 1) of a prime divides s the conditions on n can be improved over that presented in theorem 1. We shall only treat the case where $p^2 | s$ (with p a prime) although a more general result can be proved for $p^k | s$ with k arbitrary.

Theorem 2. Let $s = p^2t$ where p is a prime. A necessary condition that a perfect s-sequence exist for an integer n is

$$n \equiv -1, 0, 1, 2, \cdots, \text{ or } p - 2 \pmod{p^3} .$$

Proof. Let the integer i (with $1 \le i \le n$) occupy positions $c_1, \cdots, c_s$ in a perfect s-sequence for n. Then

$$c_j = c_1 + (j-1)(i+1) \qquad j = 1, \cdots, s .$$

We consider three categories for the integer i as follows:

I.) For the

$$n - \left[\frac{n+1}{p}\right]$$

integers i with $i + 1 \not\equiv 0 \pmod p$ the positions $c_1, \cdots, c_s$ range over the residue classes $(\bmod\ p^2)$ in such a manner that each residue class occurs exactly t times

II.) For the

$$\left[\frac{n+1}{p}\right] - \left[\frac{n+1}{p^2}\right]$$

integers i with $i + 1 \equiv 0 \pmod p$ and $i + 1 \not\equiv 0 \pmod{p^2}$ the positions $c_1, \cdots, c_s$ range over the residue classes $c_1, c_1 + p, \cdots, c_1 + (p - 1)p \pmod{p^2}$ in a manner whereby each such residue occurs exactly pt times. We let N(r) for $r = 0, 1, \cdots, p - 1$ be the number of i in this category with $c_1 \equiv r \pmod p$.

III.) For the

$$\left[\frac{n+1}{p^2}\right]$$

integers with $i + 1 \equiv 0 \pmod{p^2}$ the positions $c_1, \cdots, c_s$ all belong to the same residue class $\pmod{p^2}$.

We let M(q) for $q = 0, 1, \cdots, p^2 - 1$ be the number of i in this category with $c_1 \equiv q \pmod{p^2}$.

Letting q be a residue of p^2 with $q \equiv r \pmod p$, the number of positions in the s-sequence for n that are congruent to $q \pmod{p^2}$ is nt. Thus

$$nt = t\left\{n - \left[\frac{n+1}{p}\right]\right\} + ptN(r) + p^2tM(q)$$

or

$$p^2M(q) = \left[\frac{n+1}{p}\right] - pN(r) .$$

The latter implies that M(q) is identical for all residues q of p^2 having the common reduced residue $r \pmod p$. Letting L(r) denote this identical value,

$$\left[\frac{n+1}{p^2}\right] = \sum_{q=0}^{p^2-1} M(q) = p \sum_{r=0}^{p-1} L(r) .$$

hence, p divides

$$\left[\frac{n+1}{p^2}\right] .$$

But from theorem 1, $n + 1 = p^2d + e$ where $e = 0, 1, 2, \cdots,$ or $p - 1$, hence $d = pd'$ and $n + 1 = p^3d' + e$ which is the desired result.

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Continued from p. 10

18	1	$\frac{1}{dF_{n-1}} > \frac{1}{dF_{n-1}} \ge \frac{1}{bd} = \left\lvert\frac{a}{b} - \frac{c}{d}\right\rvert > \left\lvert\frac{F_n}{F_{n-1}} - \frac{c}{d}\right\rvert > \frac{1}{dF_{n-1}}$	$\frac{1}{dF_{n-1}} \ge \frac{1}{bd} = \left\lvert\frac{a}{b} - \frac{c}{d}\right\rvert \ge\; > \left\lvert\frac{F_n}{F_{n-1}} - \frac{c}{d}\right\rvert \ge \frac{1}{dF_{n-1}}$
18	-7	$\frac{c}{d} - \frac{F_{n+1}}{F_n} \ge \frac{F_{n+1}}{F_n} \ge \frac{F_{n+2}}{F_{n+1}}$	$\frac{c}{d} - \frac{F_{n+1}}{F_n} + \frac{F_{n+1}}{F_n} - \frac{F_{n+2}}{F_{n+1}}$

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