LAH NUMBERS FOR R-POLYNOMIALS

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1. INTRODUCTION

According to [1], [2], and [3], given two sequences of polynomials, $P_1(x, n)$ and $P_2(x, n)$, $n = 0, 1, 2, \cdots$.

$$P_k(x, n) = \sum_{m=0}^{n} C_{k,n}^{m} x^m, \qquad k = 1, 2, \tag{1}$$

$$C_{k,n}^{m} = 0, \quad \text{for } n < m, \ m < 0, \ n < 0, \tag{1a}$$

and the inverse expansion

$$x^n = \sum_{m=0}^{n} D_{k,n}^{m} P_k(x,m) \qquad k = 1, 2, \tag{2}$$

$$D_{k,n}^{m} = 0, \quad \text{for } n < m, \ m < 0, \ n < 0, \tag{2a}$$

the coefficients $C_{k,n}^{m}$ and $D_{k,n}^{m}$ are called respectively <u>Generalized Stirling Numbers of First and Second Kind</u> of the polynomials $P_k(x, n)$. Examples of such numbers can be found in [3], [4], and [5].

Let then

$$P_k(x, n) = \sum_{m=0}^{n} L_{k,h,n}^{m} P_h(x,m), \qquad k, h = 1, 2, \ k \neq h, \ n = 0, 1, 2, \cdots \tag{3}$$

$$L_{k,h,n}^{m} = 0, \quad \text{for } n < m, \ m < 0, \ n < 0 \ . \tag{3a}$$

More explicitly,

$$P_k(x,n) = \sum_{s=0}^{n} C_{k,n}^{s} x^s = \sum_{s=0}^{n} C_{k,n}^{s} \sum_{m=0}^{s} D_{h,s}^{m} P_h(x,m) ,$$

$$= \sum_{s=0}^{n} C_{k,n}^{s} \sum_{m=0}^{n} D_{h,s}^{m} P_h(x,m)$$

$$= \sum_{m=0}^{n} P_h(x,m) \sum_{s=m}^{n} C_{k,n}^{s} D_{h,s}^{m} ,$$

so that

(3b) $$L_{k,h,n}^{m} = \sum_{s=m}^{n} C_{k,n}^{s} D_{h,s}^{m} .$$

The coefficients $L_{k,h,n}^{m}$ are called Generalized Lah Numbers for the two sequences of polynomials P_k and P_h, $k \neq h$, $k, h = 1, 2$.

2. QUASI-ORTHOGONALITY

Under the conditions stated, the generalized Stirling numbers of first and second kind for a given sequence of polynomials $P_k(x,n)$ are said to quasi-orthogonal to each other (cf. [3] if

(4) $$\sum_{m=s}^{n} D_{k,n}^{m} C_{k,m}^{s} = \delta_n^s .$$

This result is proved in [3] for both the Q- and R-polynomials, but since the proof does not use the structure of the polynomials it is true for any sequence of polynomials as defined by (1).

Similarly the generalized Lah numbers for two sequences of polynomials P_k and P_h are quasi-orthogonal to the generalized Lah numbers for the sequences of polynomials P_h and P_k, i.e.,

$$(5) \qquad \sum_{m=s}^{n} L_{k,h,n}^{m} L_{h,k,m}^{s} = \delta_n^s .$$

This result is proved in [2] for the Q-polynomials, but here again the proof does not use the structure of the polynomials, thus is valid for any two sequences of polynomials as defined by (1).

3. RECALL ABOUT R-POLYNOMIALS

In [2] we have studied the generalized Lah numbers for two sequences of Q-polynomials. We shall now study the same for R-polynomials as defined in [3], i.e.,

$$(6) \qquad R(x,n) = \sum_{m=0}^{n} C_n^m x^m$$

$$(7) \qquad R(x,n+1) = [K(n+1) + L(n+1)x]R(x,n) + \sum_{m=0}^{n} [M(m+1) + N(m+2)x] C_n^m x^m ,$$

$$(8) \qquad R(x,0) = K(0)$$

$$(9) \qquad x^n = \sum_{m=0}^{n} D_n^m R(x,m) .$$

In order to simplify the results in [3] it was assumed that $L = 1$. Letting $N(n+1) + 1 = P(n)$ it was proved that the numbers C_n^m and D_n^m satisfy the recurrence relations

(10) $$C_n^m = [K(n) + M(m+1)]C_{n-1}^m + P(m)C_{n-1}^{m-1}$$

(11) $$D_n^m = -[K(m+1) + M(n)]D_{n-1}^m/P(n) + D_{n-1}^{m-1}/P(n) .$$

In the following we shall consider two sets of R-polynomials $R_1(x, n)$ and $R_2(x, n)$ and the corresponding generalized Stirling numbers $\{C_{1,n}^m, C_{2,n}^m\}$ and $\{D_{1,n}^m, D_{2,n}^m\}$ which all satisfy the conditions given in sections 1 and 2. The generalized Lah-numbers for the two sequences are $L_{1,2,n}^m$ and $L_{2,1,n}^m$. They satisfy conditions (3a), (3b), and (5). We shall assume that $L_1(n) = L_2(n) = 1$.

4. RECURRENCE RELATIONS

According to relations (6) through (9) we can write

$$R_2(x, n+1) = \sum_{s=0}^{n+1} C_{2,n+1}^m x^m$$

$$= [K_2(n+1) + x]R_2(x,n) + \sum_{s=0}^{n} [M_2(s+1) + N_2(s+2)x]C_{2,n}^s x^s ,$$

and, according to the definition of the generalized Lah-numbers,

$$R_2(x, n+1) = \sum_{m=0}^{n+1} L_{2,1,n+1}^m R_1(x, m) ,$$

so that

(12) $$\sum_{m=0}^{n+1} L_{2,1,n+1}^m R_1(x, m) = K_2(n+1) \sum_{m=0}^{n} L_{2,1,n}^m R_1(x,m)$$

$$+ \sum_{m=0}^{n} L_{2,1,n}^m xR_1(x, m)$$

$$+ \sum_{s=0}^{n} M_2(s+1)C_{2,n}^s \sum_{m=0}^{s} D_{1,s}^m R_1(x, m)$$

$$+ \sum_{s=0}^{n} N_2(s+2)C_{2,n}^s \sum_{m=0}^{s+1} D_{1,s+1}^m R_1(x, m) .$$

On the other hand we have

$$R_1(x, m+1) = [K_1(m+1) + x]R_1(x, m) + \sum_{p=0}^{m} [M_1(p+1) + N_1(p+2)x]C_{1,m}^{p} x^p$$

thus

$$xR_1(x, m) = R_1(x, m+1) - K_1(m+1)R_1(x, m) - \sum_{p=0}^{m} [M_1(p+1) + N_1(p+2)x]C_{1,m}^{p} x^p. \tag{13}$$

Substituting (13) into (12), and reorganizing the last two terms with the help of (1a) and (2a), we obtain

$$\sum_{m=0}^{n+1} L_{2,1,n+1}^{m} R_1(x, m) = K_2(n+1) \sum_{m=0}^{n} L_{2,1,n}^{m} R_1(x, m)$$

$$+ \sum_{m=0}^{n} L_{2,1,n}^{m} \left[R_1(x, m+1) - K_1(m+1)R_1(x, m) - \sum_{p=0}^{m} [M_1(p+1) + N_1(p+2)x]C_{1,m}^{p} x^p \right]$$

$$+ \sum_{m=0}^{n} R_1(x, m) \sum_{s=m}^{n} M_2(s+1)C_{2,n}^{s} D_{1,s}^{m}$$

$$+ \sum_{m=0}^{n+1} R_1(x, m) \sum_{s=m-1}^{n} N_2(s+2)C_{2,n}^{s} D_{1,s+1}^{m},$$

or, interchanging the indices m and s,

$$(14)\qquad \sum_{m=0}^{n+1} L_{2,1,n+1}^{m} R_1(x,m) = K_2(n+1)\sum_{m=0}^{n} L_{2,1,n}^{m} R_1(x,m)$$

$$+ \sum_{m=0}^{n} L_{2,1,n}^{m} R_1(x,m+1) - \sum_{m=0}^{n} L_{2,1,n}^{m} K_1(m+1) R_1(x,m)$$

$$- \sum_{s=0}^{n} L_{2,1,n}^{s} \sum_{p=0}^{s} M_1(p+1) C_{1,s}^{p} \sum_{m=0}^{p} D_{1,p}^{m} R_1(x,m)$$

$$- \sum_{s=0}^{n} L_{2,1,n}^{s} \sum_{p=0}^{s} N_1(p+2) C_{1,s}^{p} \sum_{m=0}^{p+1} D_{1,p+1}^{m} R_1(x,m)$$

$$+ \sum_{m=0}^{n} R_1(x,m) \sum_{s=m}^{n} M_2(s+1) C_{2,n}^{s} D_{1,s}^{m} + \sum_{m=0}^{n+1} R_1(x,m) \sum_{s=m-1}^{n} N_2(s+2) C_{2,n}^{s} D_{1,s+1}^{m}$$

The fourth and fifth quantities on the right-hand side of (14) can be written as follows:

$$(15)\qquad \sum_{s=0}^{n} L_{2,1,n}^{s} \sum_{p=0}^{s} M_1(p+1) C_{1,s}^{p} \sum_{m=0}^{p} D_{1,p}^{m} R_1(x,m)$$

$$= \sum_{m=0}^{n} R_1(x,m) \sum_{s=0}^{n} L_{2,1,n}^{s} \sum_{p=m}^{s} M_1(p+1) C_{1,s}^{p} D_{1,p}^{m},$$

$$(16)\qquad \sum_{s=0}^{n} L_{2,1,n}^{s} \sum_{p=0}^{s} N_1(p+2) C_{1,s}^{p} \sum_{m=0}^{p+1} D_{1,p+1}^{m} R_1(x,m)$$

$$= \sum_{m=0}^{n+1} R_1(x,m) \sum_{s=0}^{n} L_{2,1,n}^{s} \sum_{p=m-1}^{s} N_1(p+2) C_{1,s}^{p} D_{1,p+1}^{m}$$

Substituting (15) and (16) into (14) we obtain by equating the coefficients of $R_1(x,m)$

$$(17)\qquad L_{2,1,n+1}^{m} = K_2(n+1) - K_1(m+1)\ L_{2,1,n}^{m} + L_{2,1,n}^{m-1}$$

$$- \sum_{s=m-1}^{n} L_{2,1,n}^{s} \sum_{p=m}^{s} M_1(p+1)C_{1,s}^{p}D_{1,p}^{m} + \sum_{p=m-1}^{s} N_1(p+2)C_{1,s}^{p}\,D_{1,p+1}^{m}$$

$$+ \sum_{s=m}^{n} M_2(s+1)C_{2,n}^{s}\,D_{1,s}^{m} + \sum_{s=m-1}^{n} N_2(s+2)C_{1,n}^{s}D_{1,s+1}^{m} ,$$

or, changing n into $n - 1$,

$$(18)\qquad L_{2,1,n}^{m} = K_2(n) - K_1(m+1)\ L_{2,1,n-1}^{m} + L_{2,1,n-1}^{m}$$

$$- \sum_{s=m-1}^{n-1} L_{2,1,n-1}^{s} \sum_{p=m}^{s} M_1(p+1)C_{1,s}^{p}D_{1,p}^{m} + \sum_{p=m-1}^{s} N_1(p+2)C_{1,s}^{p}D_{1,p+1}^{m}$$

$$+ \sum_{s=m}^{n-1} M_2(s+1)C_{2,n-1}^{s}D_{1,s}^{m} + \sum_{s=m-1}^{n-1} N_2(s+2)C_{2,n-1}^{s}\,D_{1,s+1}^{m}$$

Relation (18) is the recurrence relation for the generalized numbers $L_{2,1,n}^{m}$. A similar relation for the Lah-numbers $L_{1,2,n}^{m}$ will be obtained by interchanging the indices 1 and 2 in (18).

5. EXAMPLE

We illustrate by the following example based on examples I and II of section 5 of [3]. Thus:

$$K_1(\alpha) = \alpha + 1,\quad M_1(\alpha) = (\alpha - 1)^2,\quad N_1(\alpha) = 0,\quad K_2(\alpha) = \alpha,\quad M_2(\alpha) = \alpha,$$
$$N_2(\alpha) = \alpha ,$$

where the index 1 corresponds to example I and the index 2 to example II of section 5 of [3]. The numerical values of $C_{1,n}^{m}$ are those of C_n^m, of $D_{1,n}^{m}$ those of D_n^m of example I, while $C_{2,n}^{m}$ and $D_{2,n}^{m}$ those of C_n^m and D_n^m of example II. Under these conditions we obtain the following for $L_{2,1,n}^{m}$:

$$L_{2,1,n}^{m} = (n - m - 2)L_{2,1,n-1}^{m} + L_{2,1,n-1}^{m-1} - \sum_{s=m-1}^{n-1} L_{2,1,n-1}^{s}\left[\sum_{p=m}^{s} p^2 C_{1,s}^{p} D_{1,p}^{m}\right] + \sum_{s=m}^{n-1}(s+1)C_{2,n-1}^{s}D_{1,s}^{m} + \sum_{s=m-1}^{n-1}(s+2)C_{2,n-1}^{s}D_{1,s+1}^{m} . \tag{19}$$

m=	0	1	2	3	4
n:					
0	1				
1	-4	3			
2	42	-54	12		
3	-1488	2124	-696	60	
4	99680	-170640	67440	-8880	360

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