ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-166 Proposed by H. H. Ferns, Victoria, B. C., Canada.

Prove the identity

$$F_{2mn} = \begin{cases} \displaystyle\sum_{i=1}^{n} \binom{n}{i} L_m^i F_{mi} & \text{if } m \text{ is odd} \\ \displaystyle\sum_{i=1}^{n} (-1)^{n+i} L_m^i F_{mi} & \text{if } m \text{ is even} \end{cases},$$

where F_n and L_n are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively.

H-167 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$S_k = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+k}} .$$

Show that, for $k \geq 0$,

$$\text{(A)} \qquad F_{2k+2} S_{2k+2} = k + 1 - \sum_{n=1}^{2k} \frac{k - [\frac{1}{2}(n-1)]}{F_n F_{n+2}} ,$$

(B) $$F_{2k+1}S_{2k+1} = S_1 - k + \sum_{n=0}^{2k-1} \frac{k - \left[\frac{n}{2}\right]}{F_n F_{n+2}} ,$$

where $[a]$ denotes the greatest integer function.

Special cases of (A) and (B) have been proved by Brother Alfred Brousseau, "Summation of Infinite Fibonacci Series," Fibonacci Quarterly, Vol. 7, No. 2, April, 1969, pp. 143-168.

H-168 Proposed by David A. Klarner, University of Alberta, Edmonton, Alberta, Canada.

If

$$a_{ij} = \binom{i + j - 2}{i - 1}$$

for $i, j = 1, 2, \cdots, n$, show that $\det\{a_{ij}\} = 1$.

SOLUTIONS

GENERALIZE

H-137 Proposed by J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania.

GENERALIZED FORM OF H-70: Consider the set S consisting of the first N positive integers and choose a fixed integer k satisfying $0 < k \leq N$. How many different subsets A of S (including the empty subset) can be formed with the property that $a' - a'' \neq k$ for any two elements a', a'' of A: that is, the integers i and $k + k$ do not both appear in A for any $i = 1, 2, \cdots, N - k$.

Solution by the Proposer.

Let $N \equiv r \pmod k$ so that $N = tk + r$ with t a positive integer and $0 \leq r \leq k - 1$.

Each subset A of S can be made to correspond to a binary sequence $(\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_N)$ of N terms by the rule that $\alpha_i = 1$ if $i \in A$ and $\alpha_i = 0$ if $i \notin A$. For a given subset A and its corresponding binary sequence $(\alpha_1, \alpha_2, \cdots, \alpha_N)$, define k binary sequences as follows:

$$
\begin{aligned}
A_1 &= (\alpha_1, \alpha_{1+k}, \alpha_{1+2k}, \cdots, \alpha_{1+tk}) \\
A_2 &= (\alpha_2, \alpha_{2+k}, \alpha_{2+2k}, \cdots, \alpha_{2+tk}) \\
&\vdots \\
A_r &= (\alpha_r, \alpha_{r+k}, \alpha_{r+2k}, \cdots, \alpha_{r+tk}) \\
A_{r+1} &= (\alpha_{r+1}, \alpha_{r+1+k}, \alpha_{r+1+2k}, \cdots, \alpha_{r+1+(t-1)k}) \\
&\vdots \\
A_k &= (\alpha_k, \alpha_{2k}, \alpha_{3k}, \cdots, \alpha_{tk}) \qquad .
\end{aligned}
$$

Then the subset A corresponding to $(\alpha_1, \alpha_2, \cdots, \alpha_N)$ satisfies the given constraint if and only if each A_m independently for $m = 1,2,\cdots,k$ is a binary sequence without consecutive 1's. But it is well known that the total number of binary sequences of length n without consecutive 1^s is F_{n+2}. Since each of the r sequences $A_1,\cdots,A_r$ has length $t+1$ and each of the remaining $k - r$ sequences $A_{r+1},\cdots,A_k$ has length t, it follows that the total number of subsets with the required property is $F^r_{r+3}\,F^{k-r}_{t-2}$.

Also solved by M. Yoder.

FIBONOMIALS

H-138 Proposed by George E. Andrews, Pennsylvania State University, University Park, Pennsylvania.

If F_n denotes the sequence of polynomials $F_1 = F_2 = 1$, $F_n = F_{n-1} + x^{n-2}F_{n-2}$, prove that $1 + x + x^2 + \cdots + x^{p-1}$ divides F_{p+1} for any prime $p = \pm 2 \pmod 5$.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Let $\Phi_n(x)$ denote the cyclotomic polynomial:

$$\Phi_n(x) = \prod_{rs=n} (x^r - 1)^{\mu(s)} ,$$

where $\mu(s)$ is the Mobius function. We shall prove that F_{n+1} is divisible by $\Phi_n(x)$ if and only if $n \equiv \pm 2 \pmod 5$, where n is an arbitrary positive integer (not necessarily prime). Indeed, we obtain the residue of F_{n+1} (mod $\Phi_n(x)$) for all n. In particular, we find that

$$F_{n+1} \equiv 1 \pmod{\Phi_n(x)}$$

when $n \equiv \pm 1 \pmod{10}$.

I. Schur (Berliner Sitzungsberichte (1917), pp. 302-321) has proved that if

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + x^n F_n \quad (n \geq 1),$$

then

$$(1) \qquad F_{n+1} = \sum_{k=-r}^{r} (-1)^k x^{\frac{1}{2}k(5k-1)} \begin{bmatrix} n \\ e(k) \end{bmatrix},$$

where

$$e(k) = \left[\tfrac{1}{2}(n + 5k)\right], \qquad r = \left[\tfrac{1}{5}(n + 2)\right]$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(1 - x^n)(1 - x^{n-1}) \cdots (1 - x^{n-k+1})}{(1 - x)(1 - x^2) \cdots (1 - x^k)} & (0 \leq k \leq n), \\ 0 & (\text{otherwise}). \end{cases}$$

$\begin{bmatrix} n \\ k \end{bmatrix}$ is a polynomial in x with positive integral coefficients: also it is evident from the definition that for $1 \leq k \leq n$, $\begin{bmatrix} n \\ k \end{bmatrix}$ is divisible by the cyclotomic polynomial $\Phi_n(x)$.

Thus (1) implies

$$(2) \quad F_{n+1} \equiv (-1)^r x^{\frac{1}{2}r(5r-1)} \begin{bmatrix} n \\ e(r) \end{bmatrix} + (-1)^r x^{\frac{1}{2}r(5r+1)} \begin{bmatrix} n \\ e(-r) \end{bmatrix} \pmod{\Phi_n(x)}.$$

The following table is easily verified.

n	r	e(r)	e(-r)
10 m	2 m	10 m	0
10 m + 1	2 m	10 m	0
10 m + 2	2 m	10 m + 1	1
10 m + 3	2 m + 1	10 m + 4	-1
10 m + 4	2 m + 1	10 m + 4	-1
10 m + 5	2 m + 1	10 m + 5	0
10 m + 6	2 m + 1	10 m + 5	0
10 m + 7	2 m + 1	10 m + 6	1
10 m + 8	2 m + 2	10 m + 9	-1
10 m + 9	2 m + 2	10 m + 9	-1

Therefore, making use of (2), we get the following values for the residue of F_{n+1} (mod $\Phi_n(x)$):

n	residue of F_{n+1} (mod $\Phi_n(x)$)
10 m	$x^{m(10m-1)} + x^{m(10m+1)} \equiv x^{9m} + x^{m}$
10 m + 1	$x^{m(10m+1)} \equiv 1$
10 m + 2	0
10 m + 3	0
10 m + 4	$-x^{(2m+1)(5m+2)} \equiv -x^{5m+2}$
10 m + 5	$-x^{(2m+1)(5m+2)} - x^{(2m+1)(5m+3)} \equiv -x^{4m+2} - x^{6m+3}$
10 m + 6	$-x^{(2m+1)(5m+3)} \equiv -x^{5m+3}$
10 m + 7	0
10 m + 8	0
10 m + 9	$x^{(m+1)(10m+9)} \equiv 1$

As a check, we compute F_{n+1}, $2 \leq n \leq 10$, and the corresponding residues,

n	F_{n+1}	residue (mod Φ_n)
2	$1 + x$	0
3	$1 + x + x^2$	0
4	$1 + x + x^2 + x^3 + x^4$	$1 \equiv -x^2$
5	$1 + x + x^2 + x^3 + 2x^4 + x^5 + x^6$	$-x^2 - x^3$
6	$1 + x + x^2 + x^3 + 2x^4 + 2x^5 + 2x^6 + x^7 + x^8 + x^9$	$1 \equiv -x^3$
7	$1 + x + x^2 + x^3 + 2x^4 + 2x^5 + 3x^6 + 2x^7 + 2x^8 + 2x^9 + 2x^{10} + x^{11} + x^{12}$	0
8	$1 + x + x^2 + x^3 + 2x^4 + 2x^5 + 3x^6 + 3x^7 + 3x^8 + 3x^9 + 3x^{10} + 3x^{11} + 3x^{12} + 2x^{13} + x^{14} + x^{15} + x^{16}$	0
9	$1 + x + x^2 + x^3 + 2x^4 + 2x^5 + 3x^6 + 3x^7 + 4x^8 + 4x^9 + 4x^{10} + 4x^{11} + 5x^{12} + 4x^{14} + 3x^{15} + 3x^{16} + 2x^{17} + 2x^{18} + x^{19} + x^{20}$	1

Remarks. 1. If we use the fuller notation $F_n(x)$ in place of F_n and ϵ denotes a primitive n^{th} root of unity, then the statement $F_{n+1}(x)$ is divisible by $\Phi_n(x)$ is equivalent to $F_{n+1}(\epsilon) = 0$. Using the recurrence for F_n, it is not difficult to show that, for n odd,

$$F_{n+1}(\epsilon) = \left| F_{\frac{1}{2}(n+3)}(\epsilon)\right|^2 - \left| F_{\frac{1}{2}(n-1)}(\epsilon)\right|^2 ,$$

while for n even,

$$F_{n+1}(\epsilon) = \left|F_{k+1}(\epsilon)\right|^2 + \epsilon^{-k}\left|F_k(\epsilon)\right|^2 \qquad (n = 2k) .$$

2. In the next place, it follows from the recurrence that

$$\sum_{n=0}^{\infty} F_{n+1}a^n = \sum_{k=0}^{\infty} \frac{a^{2k}x^{k^2}}{(a)_k} , \tag{3}$$

where

$$(a)_k = (1 - a)(1 - ax) \cdots (1 - ax^k) .$$

Since

$$\frac{1}{(a)_k} = \sum_{r=0}^{\infty} \begin{bmatrix} k + r \\ r \end{bmatrix} a^r ,$$

we get

$$F_{n+1} = \sum_{2k \le n} \begin{bmatrix} n - k \\ k \end{bmatrix} x^{k^2} .$$

If we take $a = x$ in (3), we get

$$1 + \sum_{n=1}^{\infty} F_n x^n = \sum_{k=0}^{\infty} \frac{x^{k^2}}{(x)_k} = \prod_{n=0}^{\infty} (1 - x^{5n+1})^{-1} (1 - x^{5n+4})^{-1} ,$$

by the first Roger-Ramanujan identity (see, for example, Hardy and Wright, Introduction to the Theory of Numbers, Oxford, 1954, p. 290).

Incidentally, if

$$G_1 = G_2 = 1, \quad G_{n+1} = G_n + x^n G_{n-1} \qquad (n > 1) ,$$

then we have

$$\sum_{n=0}^{\infty} G_{n+1} a^n = \sum_{k=0}^{\infty} \frac{a^{2k} x^{k^2+k}}{(a)_k} , \tag{4}$$

and

$$G_{n+1} = \sum_{2k \le n} \begin{bmatrix} n - k \\ k \end{bmatrix} x^{k^2+k} .$$

If we take $a = x$ in (4), we get

$$1 + G_n x^n = \sum_{k=0}^{\infty} \frac{x^{k^2+k}}{(x)_k} = \prod_{n=0}^{\infty} (1 - x^{5n+2})^{-1} (1 - x^{5n+3})^{-1}$$

by the second Rogers-Ramanujan identity.

INTEGRITY

H-140 Proposed by Douglas Lind, University of Virginia, Charlottesville, Virginia.

For a positive integer m, let $\alpha = \alpha(m)$ be the least positive integer such that $F_\alpha = 0 \pmod m$. Show that the highest power of a prime p dividing $F_1F_2\cdots F_n$ is

$$\sum_{k=1}^{\infty} \left[\frac{n}{\alpha(p^k)}\right]$$

where $[x]$ denotes the greatest integer contained in x. Using this, show that the Fibonacci binomial coefficients

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{F_m F_{m-1} \cdots F_{m-r+1}}{F_1 F_2 \cdots F_r} \qquad (r > 0)$$

are integers.

Solution by the Proposer.

It is known [D. D. Wall, "Fibonacci Series Modulo m," Amer. Math. Monthly, 67 (1960), 525-532] that $F_r \equiv 0 \pmod m$ if and only if $r \equiv 0 \pmod{\alpha(m)}$. Then the number of F_r with $r \le n$ which are exactly divisible by p^k is $[n/\alpha(p^k)]$, establishing the first part. Note that $\alpha(p^k) \to \infty$ as $k \to \infty$, so for fixed p this is actually a finite sum.

Now let $(m) = F_1F_2 \cdots F_m$. Then

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{(m)!}{(r)!(m - r)!} .$$

It suffices to show that for any prime p, the highest power of p dividing the numerator is not less than that dividing the denominator. By the first part, this is equivalent to

$$(\star) \qquad \sum_{k=1}^{\infty}\left[\frac{m}{\alpha(p^k)}\right] \geq \sum_{k=1}^{\infty}\left[\frac{r}{\alpha(p^k)}\right] + \sum_{k=1}^{\infty}\left[\frac{m-r}{\alpha(p^k)}\right] .$$

But the elementary inequality $[x + y] \geq [x] + [y]$ shows that

$$\left[\frac{m}{\alpha}\right] \geq \left[\frac{r}{\alpha}\right] + \left[\frac{m-r}{\alpha}\right] ,$$

implying $(\star)$ and the result.

Also solved by M. Yoder.

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[Continued from page 30.]

6. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, 1930, fourth ed., 1960.
7. O. Wyler, "On Second-Order Recurrences," Amer. Math. Monthly, 72, pp. 500-506, May, 1965,
8. D. D. Wall, "Fibonacci Series Modulo m," Amer. Math. Monthly, 67, pp. 525-532 (June 1960).
9. R. P. Backstrom, "On the Determination of the Zeros of the Fibonacci Sequence," Fibonacci Quarterly, Vol. 5, pp. 313-322, December, 1966.

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