RECREATIONAL MATHEMATICS

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ANGLE MULTISECTION BY PARALLEL STRAIGHTEDGES

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Trisection of an arbitrary angle is, of course, impossible by means of compass and unmarked straightedge alone, but the attempt to do this by other means has fascinated mathematicians since the time of the ancient Greeks.

The simplest trisection is probably Archimedes' famous paper strip construction. This method involves using a straightedge with two marks, known as an application of the "insertion principle." It is illustrated in Fig. 1.

The angle to be trisected here is θ. A circle of arbitrary radius r is drawn whose center O is the vertex of the given angle. The sides of the angle intersect the circle at points A and B. $\overrightarrow{BO}$ is extended. A segment of length r is marked on a straightedge or paper strip. The edge is placed so that it passes through A and so that one endpoint of the marked segment intersects the circle at C and the other endpoint falls at D on $\overrightarrow{BO}$, outside the circle. Then $m\angle CDO$, here marked a, is one-third $m\theta$. The proof is easily seen: $\triangle OCD$, having two sides of length r, is isosceles, so that $m\angle COD = m\angle CDO = a$. By the exterior angle theorem, $m\angle ACO = m\angle CAO = 2a$, since $\angle OAC$ is also isosceles. The given angle θ forms an exterior angle of $\triangle OAD$. Thus $m\theta = 2a + a = 3a$.

A solution of the problem using parallel straightedges, and a generalization are given here.

In Fig. 2, let the angle to be trisected be θ; a circle of arbitrary radius r is drawn with center at vertex O. A and B are the intersection points of the sides of the angle and the circle. K and F represent the two parallel straightedges; K passes through point A and F passes through center O. C and E are points where each straightedge intersects the circle. Mark point D on straightedge K such that CD = r. Adjust the straightedges so that $\overline{CE} \perp \overrightarrow{BO}$. When this occurs, point D will fall upon $\overrightarrow{BO}$, for the reason that □OCDE is a rhombus. To prove it, let D' be the point of intersection of K and $\overrightarrow{BO}$. ΔOPC and ΔOEP are $\cong$ by H-L; $\angle CD'P \cong \angle EOP$ by alt. int. angles; $\overline{CP} \cong \overline{EP}$ by cpctc. Therefore, $\Delta OEP = \Delta D'CP$ by SAA. It then follows that $OE = OC = CD'$, $\overline{CD'}$ parallel to $\overline{OE}$, diagonals are $\perp$, therefore □OCD'E is a rhombus, and $D = D'$. Thus $m\angle ODA = a = 1/3\, m\theta$. $\angle DOE$ and the angle vertical to it have measure a.

It is possible to broaden the scope of the previous method to certain problems of multisection — that of dividing an arbitrary angle into a given number of equal parts — combining the ideas used in the Archimedes trisection with the properties of what might be thought of as a set of "collapsing rhombuses." Specifically, it makes possible division of a given arbitrary angle into $2^n + 1$ equal parts, where n may be any positive integer. In fact, the angle may be divided into any number of parts which is a <u>divisor</u> of a number of the form $2^n + 1$ — for example, if $n = 5$, it is possible to divide the angle into 33 parts or 11 parts by taking three of the parts each equal to 1/33 of the angle. The method shown above for trisection represents the case where $n = 1$; that is, $2^n = 3$. When $n = 2, 3, 4, \cdots$, it may be seen that any given angle may be divided into $5, 9, 17, \cdots$ equal parts, respectively.

If $n = 2$ so that $2^n + 1 = 5$, we have a 5-section as shown in Fig. 3. In each case, incidentally, there appear n rhombuses — we see two here. As before the angle to be 5-sected is represented by θ, the circle is drawn with radius r and center O. An inserted length equal to r on straightedge K has endpoints C and D, with C on the circle. Straightedge F is parallel to K. K and F pass through points A and O, respectively. E is the point of intersection of F and the circle. This time □OEDC is a rhombus with diagonals $\overline{OD}$ and $\overline{CE}$ (sides $\overline{CD}$ and $\overline{OE}$ are equal in length and parallel). H is the point of intersection of diagonal $\overline{OD}$ and the circle. K is adjusted so that $\overline{HE} \perp \overrightarrow{BO}$. Now using OD as radius, O as center, draw a circle concentric to the original. The intersection of F and this circle is point M. Draw $\overline{DM}$. It can be seen that $\overline{DM} \perp \overrightarrow{BO}$. □ODGM is a rhombus so that $OD = DG$. If $m\angle DGO = a$, then $m\angle DOG = a$. By the exterior angle theorem m$\angle$ CDO = m$\angle$ COD = 2a. Now $\angle$ ACO is an exterior angle to ΔCOG, so that m$\angle$ ACO = m$\angle$ COG + m$\angle$ CGO = 3a + a = 4a = m$\angle$ OAC. The given angle, θ, is an exterior angle to ΔAOG, so that m = 4a + a = 5a. By alt. interior angles, m$\angle$ GOE = m$\angle$ DGO = a, which is the measure also of the angle vertical to $\angle$ GOE. Thus the angle θ is 5-sected.

In each case, it may be seen, as in Fig. 4, which shows a 9-section with 3 rhombuses, that a diagonal of each small rhombus becomes a side of the next larger rhombus. Each succeeding case is similar to that outlined in the 5-section, in principle. The properties of the rhombus, in particular its equal sides and perpendicular diagonals, provide the means for this interesting method whereby the chain of rhombuses could be extended to infinity.

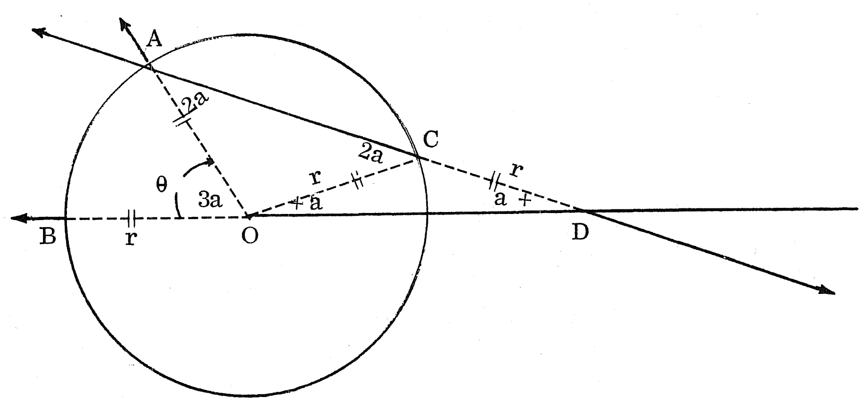

Fig. 1 Archimedes' Trisection

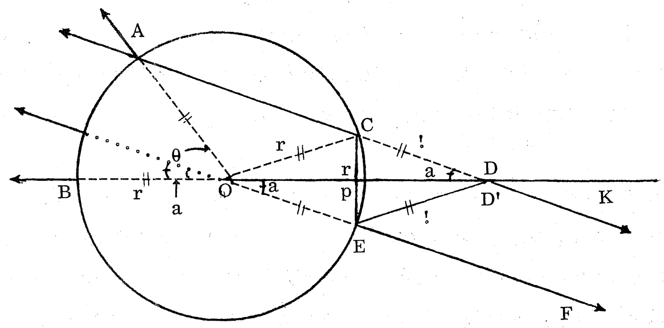

Fig. 2 Trisection Using Parallel Straightedges
! (Proved, not Given)

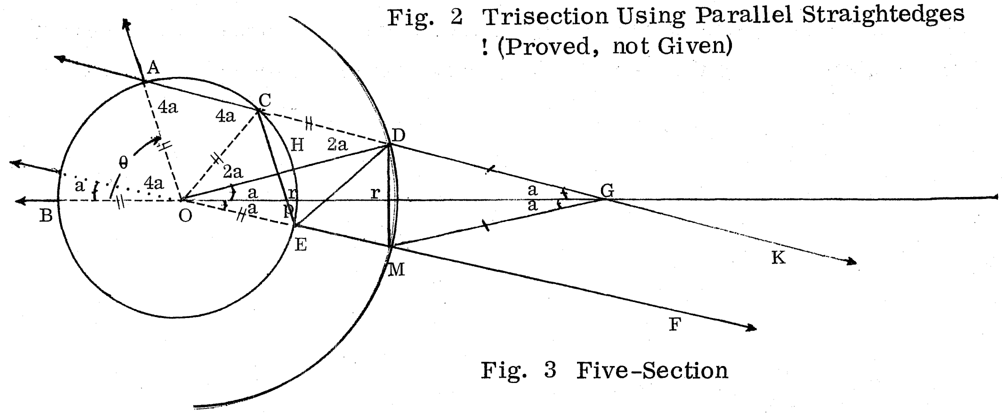

Fig. 3 Five-Section

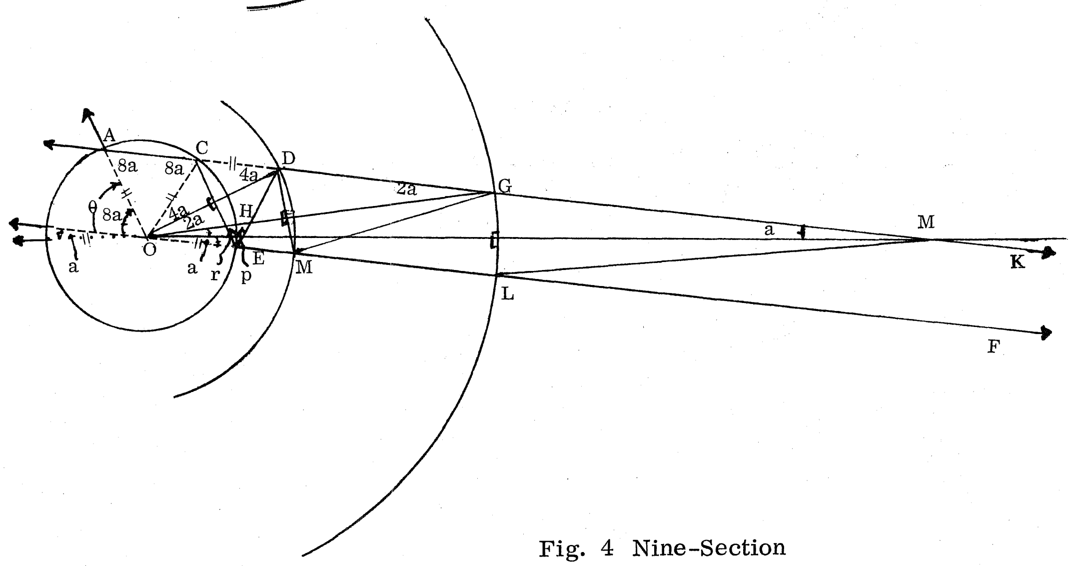

Fig. 4 Nine-Section