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Extending Zeckendorf's Theorem to a Non-constant Recurrence and the Zeckendorf Game on this Non-constant Recurrence Relation,

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Abstract

Zeckendorf's Theorem states that every positive integer can be uniquely represented as a sum of non-adjacent Fibonacci numbers, indexed from $1, 2, 3, 5, \ldots$. This has been generalized by many authors, in particular to constant coefficient fixed depth linear recurrences with positive (or in some cases non-negative) coefficients. In this work we extend this result to a recurrence with non-constant coefficients, $a_{n+1} = na_n + a_{n-1}$. The decomposition law becomes every m has a unique decomposition as $\sum s_i a_i$ with $s_i \leq i$, where if $s_i = i$ then $s_{i-1} = 0$. Similar to Zeckendorf's original proof, we use the greedy algorithm. We show that almost all the gaps between summands, as n approaches infinity, are of length zero, and give a heuristic that the distribution of the number of summands tends to a Gaussian.

Furthermore, we build a game based upon this recurrence relation, generalizing a game on the Fibonacci numbers. Given a fixed integer n and an initial decomposition of $n = na_1$, the players alternate by using moves related to the recurrence relation, and whoever moves last wins. We show that the game is finite and ends at the unique decomposition of n, and that either player can win in a two-player game. We find the strategy to attain the shortest game possible, and the length of this shortest game. Then we show that in this generalized game when there are more than three players, no player has the winning strategy. Lastly, we demonstrate how one player in the two-player game can force the game to progress to their advantage.