Thomas C. Martinez, Steven J. Miller, Clayton Mizgerd,
and Chenyang Sun
Generalizing Zeckendorf's Theorem to Homogeneous Linear Recurrences, I,
Fibonacci Quart. **60** (2022), no. 5, 222–230.

Abstract

Zeckendorf's theorem states that every positive integer can be written uniquely as the sum of nonconsecutive shifted Fibonacci numbers $\{F_n\}$, where we take $F_1 = 1$ and $F_2 = 2$. This has been generalized for any Positive Linear Recurrence Sequence (PLRS), which informally is a sequence satisfying a homogeneous linear recurrence with a positive leading coefficient and nonnegative integer coefficients. These decompositions are generalizations of base B decompositions. In this and the followup paper, we provide two approaches to investigate linear recurrences with leading coefficient zero, followed by nonnegative integer coefficients, with differences between indices relatively prime (abbreviated ZLRR). The first approach involves generalizing the definition of a legal decomposition for a PLRS found in Koloğlu, Kopp, Miller, and Wang. We prove that every positive integer N has a legal decomposition for any ZLRR using the greedy algorithm. We also show that a specific family of ZLRRs loses uniqueness of decompositions. The second approach converts a ZLRR to a PLRR that has the same growth rate. We develop the Zeroing Algorithm, a powerful helper tool for analyzing the behavior of linear recurrence sequences. We use it to prove a general result that guarantees the possibility of conversion between certain recurrences, and develop a method to quickly determine whether certain sequences diverge to $+\infty$ or $-\infty$, given any real initial values. This paper investigates the first approach.