36-tone tempered scale, so that every third member of the sequence is very nearly one of the twelve tones of our present musical scale. For perfect correspondence, such that every third tone is 100, 200, 300, etc. cents, the value of r should be 1.618261.

The usual method of constructing tempered scales is to use a ratio r which is the nth root of 2 to obtain a scale of n equidistant tones. $\sqrt[36]{2} = 1.019440644$. The ratio 1.618261 is a power of this, in fact the 25th power. It is interesting to note that 1.618... itself is not a frequency ratio that corresponds to a tone of our 12-tone scale, for it gives 833 cents, far enough from 800 to sound sharp and give discords. Other attempts to relate the Golden Ratio to musical pitch have overlooked this hard musical fact. The present discussion may serve to reinstate the Divine Proportion into the Divine Harmony.

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EXPONENTIAL GENERATION OF BASIC LINEAR IDENTITIES*

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Generalizing results of Fibonacci and Lucas numbers has been an occupation of a large number of mathematicians down through the years. Frequently, one approach taken is to first prove a result involving the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ and the Lucas sequence $\{L_n\}_{n=0}^{\infty}$ and then extend it to a result or results of special cases of the sequences $\{F_{nk+r}\}_{n=0}^{\infty}$ and $\{L_{nk+r}\}_{n=0}^{\infty}$, where k and r are fixed integers. In this paper attention is focused on deriving identities related to these latter sequences. Such results, called linear because of the subscripts, are surveyed in [1]. The exponential generating functions for these latter sequences are now shown to be most productive in deriving basic linear identities that the author believes to be new. In addition, alternate derivations of several known results will be given to show the great usefulness of these generating functions in attacking a variety of Fibonacci and Lucas problems.

Recalling the Maclaurin series expansion for e^x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and hence

$$e^{Ax} = 1 + \frac{Ax}{1!} + \frac{(Ax)^2}{2!} + \cdots = \sum_{n=0}^{\infty} A^n \frac{x^n}{n!}, \tag{1}$$

for any constant A, we note that the exponential generating functions for the first mentioned sequences are

$$\sum_{n=0}^{\infty} F_n \frac{x^n}{n!} = \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}$$

and

$$\sum_{n=0}^{\infty} L_n \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x}$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$.

The exponential generating functions of the sequences of interest in this paper are found by use of (1) to be

$$\sum_{n=0}^{\infty} F_{nk+r} \frac{x^n}{n!} = \frac{\alpha^r e^{\alpha^k x} - \beta^r e^{\beta^k x}}{\alpha - \beta} \tag{2}$$

$$\sum_{n=0}^{\infty} L_{nk+r} \frac{x^n}{n!} = \alpha^r e^{\alpha^k x} + \beta^r e^{\beta^k x} \tag{3}$$

$$\sum_{n=0}^{\infty} (-1)^n F_{nk+r} \frac{x^n}{n!} = \frac{\alpha^r e^{-\alpha^k x} - \beta^r e^{-\beta^k x}}{\alpha - \beta} \tag{4}$$

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(5) $$\sum_{n=0}^{\infty}(-1)^n L_{nk+r}\frac{x^n}{n!} = \alpha^r e^{-\alpha^k x} + \beta^r e^{-\beta^k x}$$

(6) $$\sum_{n=0}^{\infty}\alpha^n F_{nk+r}\frac{x^n}{n!} = \frac{\alpha^r e^{\alpha^{k+1}x} - \beta^r e^{\alpha\beta^k x}}{\alpha - \beta}$$

(7) $$\sum_{n=0}^{\infty}\beta^n F_{nk+r}\frac{x^n}{n!} = \frac{\alpha^r e^{\alpha^k\beta x} - \beta^r e^{\beta^{k+1}x}}{\alpha - \beta}$$

(8) $$\sum_{n=0}^{\infty}\alpha^n L_{nk+r}\frac{x^n}{n!} = \alpha^r e^{\alpha^{k+1}x} + \beta^r e^{\alpha\beta^k x}$$

(9) $$\sum_{n=0}^{\infty}\beta^n L_{nk+r}\frac{x^n}{n!} = \alpha^r e^{\alpha^k\beta x} + \beta^r e^{\beta^{k+1}x}.$$

Exponential generating functions are given a considerable workout in [2] in deriving many Fibonacci and Lucas identities.

By convoluting any pair of the above series and then equating like coefficients, a linear identity is found. To begin we convolute series (2) with itself.

$$\sum_{n=0}^{\infty}F_{nk+r}\frac{x^n}{n!}\cdot\sum_{n=0}^{\infty}F_{nk+r}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}F_{jk+r}F_{(n-j)k+r}\frac{x^n}{n!}$$

and

$$\left(\frac{\alpha^r e^{\alpha^k x} - \beta^r e^{\beta^k x}}{\alpha - \beta}\right)^2 = \frac{1}{5}\left[\left(\alpha^{2r}e^{2\alpha^k x} + \beta^{2r}e^{2\beta^k x}\right) - 2(\alpha\beta)^r e^{(\alpha^k+\beta^k)x}\right]$$

$$= \sum_{n=0}^{\infty}\frac{1}{5}\left[2^n L_{nk+2r} + 2(-1)^{r+1}L_k^n\right]\frac{x^n}{n!}.$$

Hence

(10) $$\sum_{j=0}^{n}\binom{n}{j}F_{jk+r}F_{(n-j)k+r} = \frac{1}{5}\left[2^n L_{nk+2r} + 2(-1)^{r+1}L_k^n\right].$$

The convolutions of series (3) with itself and then series (2) with (3) yield the following results:

(11) $$\sum_{j=0}^{n}\binom{n}{j}L_{jk+r}L_{(n-j)k+r} = 2^n L_{nk+2r} + 2(-1)^r L_k^n$$

(12) $$\sum_{j=0}^{n}\binom{n}{j}F_{jk+r}L_{(n-j)k+r} = 2^n F_{nk+2r}.$$

Several additional summations which reduce to simple expressions are found following the same procedure. Convolutions of (4) with (2), (4) with (3), (6) with (7), and (8) with (9), respectively, yield a representative class of the identities easily derived from the given generating functions.

$$\sum_{n=0}^{\infty}(-1)^n F_{nk+r}\frac{x^n}{n!}\cdot\sum_{n=0}^{\infty}F_{nk+r}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(-1)^j F_{jk+r}F_{(n-j)k+r}\frac{x^n}{n!}$$

and

$$\left(\frac{\alpha^r e^{-\alpha^k x} - \beta^r e^{-\beta^k x}}{\alpha - \beta}\right)\left(\frac{\alpha^r e^{\alpha^k x} - \beta^r e^{\beta^k x}}{\alpha - \beta}\right) = \frac{1}{5}\left[\left(\alpha^{2r} + \beta^{2r}\right) - (\alpha\beta)^r\left(e^{(-\alpha^k+\beta^k)x} + e^{(\alpha^k-\beta^k)x}\right)\right]$$

$$= \frac{1}{5}\left[L_{2r} + (-1)^{r+1}\left(e^{-\sqrt{5}F_k x} + e^{\sqrt{5}F_k x}\right)\right]$$

$$= \frac{1}{5}\left\{L_{2r} + (-1)^{r+1}\sum_{n=0}^{\infty}5^{n/2}F_k^n[(-1)^n + 1]\frac{x^n}{n!}\right\}.$$

By equating like coefficients, we have

(13) $$\sum_{j=0}^{2n}\binom{2n}{j}(-1)^j F_{jk+r}F_{(2n-j)k+r} = 2(-1)^{r+1}5^{n-1}F_k^{2n}, \text{ for } n > 0,$$

and

$$(14)\qquad \sum_{j=0}^{2n+1}\binom{2n+1}{j}(-1)^j F_{jk+r}F_{(2n-j+1)k+r} = 0, \text{ for } n \ge 0.$$

Now considering series (4) with (3), the identities

$$(15)\qquad \sum_{j=0}^{2n}\binom{2n}{j}(-1)^j F_{jk+r}L_{(2n-j)k+r} = 0, \text{ for } n > 0,$$

and

$$(16)\qquad \sum_{j=0}^{2n+1}\binom{2n+1}{j}(-1)^j F_{jk+r}L_{(2n-j+1)k+r} = 2(-1)^{r+1}5^n F_k^{2n+1}, \text{ for } n \ge 0,$$

are deduced.

Similarly, we find

$$(17)\qquad \sum_{j=0}^{n}\binom{n}{j}\alpha^j\beta^{n-j}F_{jk+r}F_{(n-j)k+r} = \frac{1}{\sqrt{5}}\left\{F_{nk+2r} + \frac{(-1)^r}{\sqrt{5}}\left[L_{k+1}^n + (-L_{k-1})^n\right]\right\}$$

and

$$(18)\qquad \sum_{j=0}^{n}\binom{n}{j}\alpha^j\beta^{n-j}L_{jk+r}L_{(n-j)k+r} = L_{nk+2r} + (-1)^r\left[L_{k+1}^n + (-L_{k-1})^n\right].$$

A direction of generalization of the given results as well as derivation of new results is to find additional generating functions. Then aided by several lemmas that simplify the exponents of e resulting from convolutions, many linear identities are found.

To generalize the given generating functions we begin with series (2). Replacing α^k by $\alpha^k F_m$ and β^k by $\beta^k F_m$ where m is a fixed nonzero integer, leads to

$$\frac{\alpha^r e^{\alpha^k F_m x} - \beta^r e^{\beta^k F_m x}}{\alpha - \beta} = \frac{\alpha^r \sum_{n=0}^{\infty}(\alpha^k F_m)^n \frac{x^n}{n!} - \beta^r \sum_{n=0}^{\infty}(\beta^k F_m)^n \frac{x^n}{n!}}{\alpha - \beta} = \sum_{n=0}^{\infty} F_m^n \frac{(\alpha^{nk+r} - \beta^{nk+r})}{\alpha - \beta}\frac{x^n}{n!},$$

and hence

$$(19)\qquad \sum_{n=0}^{\infty} F_m^n F_{nk+r}\frac{x^n}{n!} = \frac{\alpha^r e^{\alpha^k F_m x} - \beta^r e^{\beta^k F_m x}}{\alpha - \beta}.$$

Each additional generating function given is similarly derived. (Note: Letting $m = 1$, we have $F_m^n = 1$ and then are back to the original generating function.) Only three additional generalized generating functions are listed.

$$(20)\qquad \sum_{n=0}^{\infty} L_m^n F_{nk+r}\frac{x^n}{n!} = \frac{\alpha^r e^{\alpha^k L_m x} - \beta^r e^{\beta^k L_m x}}{\alpha - \beta}$$

$$(21)\qquad \sum_{n=0}^{\infty} L_m^n L_{nk+r}\frac{x^n}{n!} = \alpha^r e^{\alpha^k L_m x} + \beta^r e^{\beta^k L_m x}$$

$$(22)\qquad \sum_{n=0}^{\infty} F_m^n L_{nk+r}\frac{x^n}{n!} = \alpha^r e^{\alpha^k F_m x} + \beta^r e^{\beta^k F_m x}.$$

The Binet definition of the numbers involved proves several useful lemmas.

Lemma 1: $\alpha^k = \alpha F_k + F_{k-1}$, $\beta^k = \beta F_k + F_{k-1}$, $\alpha^k = \frac{1}{\sqrt{5}}(\alpha L_k + L_{k-1})$, and $\beta^k = -\frac{1}{\sqrt{5}}(\beta L_k + L_{k-1})$, for any integer k.

Lemma 2: $\alpha^k F_m = F_{m+k} - \beta^m F_k$, $\beta^k F_m = F_{m+k} - \alpha^m F_k$, $\alpha^k L_m = L_{m+k} + \beta^m\sqrt{5}F_k$, and $\beta^k L_m = L_{m+k} - \alpha^m\sqrt{5}F_k$, for any integers k and m.

Substitution of these results into the given generating functions yields identities of interest in themselves. For example, consider series (2) and (19). From Lemma 1, it follows that

$$\sum_{n=0}^{\infty} F_{nk+r}\frac{x^n}{n!} = \frac{\alpha^r e^{(\alpha F_k + F_{k-1})x} - \beta^r e^{(\beta F_k + F_{k-1})x}}{\alpha - \beta} = \frac{\alpha^r \sum_{n=0}^{\infty}(\alpha F_k + F_{k-1})^n \frac{x^n}{n!} - \beta^r \sum_{n=0}^{\infty}(\beta F_k + F_{k-1})^n \frac{x^n}{n!}}{\alpha - \beta}$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}F_k^j F_{j+r} F_{k-1}^{n-j}\frac{x^n}{n!},$$

which yields

$$(23) \qquad F_{nk+r} = \sum_{j=0}^{n}\binom{n}{j}F_k^j F_{j+r} F_{k-1}^{n-j}.$$

This identity has been derived by distinct approaches in [3] and [4].

$$\sum_{n=0}^{\infty} F_m^n F_{nk+r}\frac{x^n}{n!} = \frac{\alpha^r e^{\alpha^k F_m x} - \beta^r e^{\beta^k F_m x}}{\alpha - \beta} = \frac{\alpha^r e^{(F_{m+k} - \beta^m F_k)x} - \beta^r e^{(F_{m+k} - \alpha^m F_k)x}}{\alpha - \beta}$$

$$= \frac{\alpha^r \sum_{n=0}^{\infty}(F_{m+k} - \beta^m F_k)^n \frac{x^n}{n!} - \beta^r \sum_{n=0}^{\infty}(F_{m+k} - \alpha^m F_k)^n \frac{x^n}{n!}}{\alpha - \beta}$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j+r+1}F_{m+k}^j F_k^{n-j} F_{m(n-j)-r}$$

and so

$$(24) \qquad F_m^n F_{nk+r} = \sum_{j=0}^{\infty}\binom{n}{j}(-1)^{n-j+r+1}F_{m+k}^j F_k^{n-j} F_{m(n-j)-r}.$$

The corresponding Lucas number results are

$$(25) \qquad L_{2nk+r} = \frac{1}{5^n}\sum_{j=0}^{n}\binom{2n}{j}L_k^j L_{k-1}^{2n-j} L_{j+r},$$

$$(26) \qquad L_{(2n+1)k+r} = \frac{1}{5^n}\sum_{j=0}^{2n+1}\binom{2n+1}{j}L_k^j L_{k-1}^{2n-j+1} F_{j+r}, \text{ and}$$

$$(27) \qquad L_m^n L_{nk+r} = \sum_{j=0}^{n}\binom{n}{j}(-1)^r 5^{(n-j)/2} L_{m+k}^j F_j^{n-j}\left[\beta^{m(n-j)-r} + (-1)^{n-j}\alpha^{m(n-j)-r}\right].$$

An alternate approach to identities of similar form is given in [2].

Several basic identities given early in the paper are now generalized by use of generating functions (19) to (22). It is of much interest to compare the original results with their generalized form. We now consider the convolution of series (19) with (20).

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}F_m^j L_m^{n-j} F_{jk+r} F_{(n-j)k+r}\frac{x^n}{n!} = \left(\frac{\alpha^r e^{\alpha^k F_m x} - \beta^r e^{\beta^k F_m x}}{\alpha - \beta}\right)\left(\frac{\alpha^r e^{\alpha^k L_m x} - \beta^r e^{\beta^k L_m x}}{\alpha - \beta}\right)$$

$$= \sum_{n=0}^{\infty}\frac{1}{5}\left[2^n F_{m+1}^n L_{nk+2r} + \sum_{j=0}^{n}\binom{n}{j}(-1)^{jk+r+1}F_m^j L_m^{n-j} L_{(n-2j)k}\right]\frac{x^n}{n!}.$$

Hence,

$$(28) \qquad \sum_{j=0}^{n}\binom{n}{j}F_m^j L_m^{n-j} F_{jk+r} F_{(n-j)k+r} = \frac{1}{5}\left[2^n F_{m+1}^n L_{nk+r} + \sum_{j=0}^{n}\binom{n}{j}(-1)^{jk+r+1}F_m^j L_m^{n-j} L_{(n-2j)k}\right]$$

and so

$$(29) \qquad \sum_{j=0}^{n}\binom{n}{j}F_m^j L_m^{n-j}\left[F_{jk+r} F_{(n-j)k+r} + \frac{(-1)^{jk+r}}{5}L_{(n-2j)k}\right] = \frac{2^n}{5}F_{m+1}^n L_{nk+2r}.$$

Results of similar form may be derived by utilization of the other generating functions. For example, from series (19) and (21), we obtain

$$(30) \qquad \sum_{j=0}^{n}\binom{n}{j}F_m^j L_m^{n-j} F_{jk+r} L_{(n-j)k+r} = 2^n F_{m+1}^n F_{nk+2r} + \sum_{j=0}^{n}\binom{n}{j}(-1)^{jk+r+1}F_m^j L_m^{n-j} F_{(n-2j)k}$$

and

$$(31)\qquad \sum_{j=0}^{n}\binom{n}{j}F_m^jL_m^{n-j}\left[F_{jk+r}L_{(n-j)k+r} + (-1)^{jk+r}F_{(n-2j)k}\right] = 2^nF_{m+1}^nF_{nk+2r}.$$

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IDENTITIES OF A GENERALIZED FIBONACCI SEQUENCE

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The purpose of this note is to give identities of third power and above of the generalized Fibonacci sequence with nth term H_n satisfying the recurrence relation $H_n = pF_n + qF_{n-1}$ and $H_0 = q$ where F_n denotes the nth classical Fibonacci number.

We refer to the following identities of A. F. Horadam [1]:

$$(1)\qquad H_nH_{n+2} - H_{n+1}^2 = (-1)^n e$$

$$(2)\qquad H_{m+h}H_{m+k} - H_mH_{m+h+k} = (-1)^m eF_hF_k$$

$$(3)\qquad H_m = F_{k+1}H_{m-k} + F_kH_{m-k-1}$$

and also use

$$(4)\qquad H_{k+1}H_{k+2}H_{k+4}H_{k+3} = H_{k+5}^4 - e^2$$

where $e = p^2 - pq - q^2$.

Identity 1: $H_n^4 - 2H_{n+1}^3H_n - H_{n+1}^2H_n^2 + 2H_n^3H_{n+1} + H_{n+1}^4 = e^2$.

Identity 2: $H_{n+4}^4 - 4H_{n+3}^4 - 19H_{n+2}^4 - 4H_{n+1}^4 + H_n^4 = -6e^2$.

Identity 3: $H_{n+5}^4 = 5H_{n+4}^4 + 15H_{n+3}^4 - 15H_{n+2}^4 - 5H_{n+1}^4 + H_n^4$.

Identity 4: $25\sum_{k=0}^{n} H_k^4 = H_{n+3}^4 - 3H_{n+2}^4 - 22H_{n+1}^4 - H_n^4 + 6e^2(n-1) + A$

where $A = 15p^4 - 32p^3q - 12p^2q^2 + 16pq^3 + 34q^4$.

Identity 5: A. $18\sum_{k=1}^{n}(-1)^kH_k^4 = (-1)^n(H_{n+4}^4 - 6H_{n+3}^4 - 9H_{n+2}^4 + 24H_{n+1}^4 - H_n^4)$;

B. $9\sum_{k=1}^{n}(-1)^kH_k^4 = (-1)^n(-H_{n+3}^4 + 5H_{n+2}^4 + 14H_{n+1}^4 - H_n^4 - 3e^2)$.

Identity 6: $25\sum H_{k+1}H_{k+2}H_{k+4}H_{k+5} = 26H_{n+3}^4 + 22H_{n+2}^4 + 3H_{n+1}^4 - H_n^4 - C$,

where $C = 19e^2n + (66p^4 + 70p^3q + 131p^2q^2 + 146pq^3 + 47q^4)$.