116027	10457	119057	7187	122117	3677	125687
18917	113147	13457	118757	7727	124277	38747
114197	12227	119657	6947	125207	47297	89237
14867	118037	8387	124427	43457	89417	34337
77867	48197	79907	47207	83207	35267	92507
119087	72977	61967	79757	37967	92867	27107
68687	46457	77267	41897	89087	29027	103067
53507	45737	45887	84857	32237	98867	23417
73517	45557	108677	36767	94727	23327	103997
73907	44987	33647	90107	25847	103577	19727
52127	81707	93047	31727	101957	5477	107057
75377	82217	22307	95027	20147	108587	17117
25097	82847	104597	26927	106637	14057	109847
80447	42197	13757	101267	14657	112757	14387
67187	98717	113417	20357	123737	9767	123677
7547	53087	5657	106307	9137	117167	11807
48767	78437	62987	15287	117797	5987	115547
111767	8597	118247	47837	3767	120917	7607
12437	114407	6977	119687	46817	1787	120587
107717	13487	113177	7877	118907	62597	5147
10607	116177	7577	119447	4517	122957	39047

FIGURE 1c. Right-Hand Third of Square

*****

SOME EXTENSIONS OF PROPERTIES OF THE SEQUENCE OF FIBONACCI POLYNOMIALS

JOHN R. HOWELL
Hill Junior College, Hillsboro, Texas

Sequences of functions, $\langle g_n \rangle$, that satisfy the recursion formula

(1) $$g_{n+2}(x) = axg_{n+1}(x) + bg_n(x)$$

where a and b are constants, inherit many of the properties of the sequence of Fibonacci polynomials [1]. This paper is intended to present some of these extensions.

1. BASIC DEFINITIONS AND PROPERTIES

Suppose that a and b are numbers. Let R denote the set of real numbers and C denote the set of complex numbers.

Definition 1: If $V \subseteq R$, $\mathcal{S}_{(a,b)}(V) = \{\langle g_n \rangle |$. For each natural number $p, g_p: V \to C$ and $g_{p+2}(x) = axg_{p+1}(x) + bg_p(x)$ for each $x \in V\}$.

If $V_1 \subseteq V_2$, it is easy to verify that if $\langle g_n \rangle \in \mathcal{S}_{(a,b)}(V_2)$, the corresponding sequence of restrictions is an element of $\mathcal{S}_{(a,b)}(V_1)$.

Theorem 1: If $\langle g_n \rangle$ and $\langle h_n \rangle$ are members of $\mathcal{S}_{(a,b)}(V)$ and $s: V \to C$ and $t: V \to C$, then $\langle sg_n + th_n \rangle \in \mathcal{S}_{(a,b)}(V)$. The proof for Theorem 1 is a straightforward computation.

Theorem 2: If $\{\langle g_n \rangle, \langle h_n \rangle\} \subseteq \mathcal{S}_{(a,b)}(V)$, then $\langle g_n \rangle = \langle h_n \rangle$ if and only if $g_1 = h_1$ and $g_2 = h_2$.

The proof of one of the implications of Theorem 2 is an application of the definition of equality of sequences. The other implication is an easy induction proof.

The elements of $\mathcal{S}_{(a,b)}(V)$ share a common summation formula.

Theorem 3: Suppose that for each natural number p, $g_p: V \to C$. $\langle g_n \rangle \in \mathcal{S}_{(a,b)}(V)$ if and only if for each natural number p,

$$(ax + b - 1)\sum_{j=1}^{p} g_j(x) = g_{p+1}(x) + bg_p(x) + (ax - 1)g_1(x) - g_2(x).$$

Proof: If $\langle g_n \rangle \in \mathcal{S}_{(a,b)}(V)$, the summation formula can be proved by a simple inductive argument. If $\langle g_n \rangle$ is a sequence of complex-valued functions on V with the given summation formula, then for each natural number p, the identity

$$(ax + b - 1)g_{p+1}(x) = (ax + b - 1)\left[\sum_{j=1}^{p+1} g_j(x) - \sum_{j=1}^{p} g_j(x)\right]$$

can be transformed into the equation $axg_{p+1}(x) = g_{p+2}(x) - bg_p(x)$ and thus

$$\langle g_n \rangle \in \mathcal{S}_{(a,b)}(V).$$

One element of $\mathcal{S}_{(a,b)}(R)$ seems to correspond to the sequence of Fibonacci polynomials.

Definition 2: Let $W_{(a,b)} = \langle w_n \rangle$ be the element of $\mathcal{S}_{(a,b)}(R)$ defined by $w_1(x) = 1$ and $w_2(x) = ax$.

$W_{(a,b)}$ is well defined as a consequence of Theorem 2. $W_{(1,1)}$, for example, is the sequence of Fibonacci polynomials. If $a \neq 0$ and $b \neq 0$, M. N. S. Swamy's formula [2] for the Fibonacci polynomials can be modified to give the following formula:

$$w_p(x) = \sum_{j=0}^{[(p-1)/2]} \binom{p-1-j}{j} (ax)^{p-1-2j} b^j.$$

The importance of $W_{(a,b)}$ is illustrated by the following theorem, which can easily be proved by induction.

Theorem 4: Suppose $V \subseteq R$ and that $\langle g_n \rangle$ is a sequence of complex-valued functions on V. $\langle g_n \rangle \in \mathcal{S}_{(a,b)}(V)$ if and only if $g_{p+2} = bg_1 w_p + g_2 w_{p+1}$ for each natural number p.

2. THE BINET FORMS FOR $W_{(a,b)}$

Definition 3: Let $A(x) = \dfrac{ax + \sqrt{a^2x^2 + 4b}}{2}$ and $B(x) = \dfrac{ax - \sqrt{a^2x^2 + 4b}}{2}$.

Theorem 5: $\langle 1, A, A^2, A^3, \ldots \rangle$ and $\langle 1, B, B^2, B^3, \ldots \rangle$ are elements of $\mathcal{S}_{(a,b)}(R)$.

Proof: $A^2(x) = axA(x) + b$ and $B^2(x) = axB(x) + b$. Using these two facts,

$$A^{p+2}(x) = A^2(x)A^p(x) = axA(x)A^p(x) + bA^p(x) = axA^{p+1}(x) + bA^p(x)$$

and

$$B^{p+2}(x) = B^2(x)B^p(x) = axB(x)B^p(x) + bB^p(x) = axB^{p+1}(x) + bB^p(x).$$

Theorem 6: For each natural number p, $(A - B)w_p = A^p - B^p$.

Proof: For each natural number p, let $h_p = (A - B)w_p$ and $g_p = A^p - B^p$. As a consequence of Theorem 1 and Theorem 5,

$$\{\langle h_n \rangle \cdot \langle g_n \rangle\} \subseteq \mathcal{S}_{(a,b)}^{(R)}.$$

By direct computation, $h_1 = g_1$ and $h_2 = g_2$. By Theorem 2, $\langle g_n \rangle = \langle h_n \rangle$ and the result follows by equating corresponding terms.

3. MATRIX GENERATORS

Let $Q = \begin{pmatrix} ax & 1 \\ b & 0 \end{pmatrix}$

Theorem 7: If $\langle g_n \rangle \in \mathcal{S}_{(a,b)}^{(V)}$, then for each natural number p,

$$\begin{pmatrix} g_{p+2} & g_{p+1} \\ g_{p+1} & g_p \end{pmatrix} = \begin{pmatrix} g_3 & g_2 \\ g_2 & g_1 \end{pmatrix} Q^{p-1}$$

Theorem 7 can be proved with a simple induction argument. Using Theorem 7, many identities analogous to familiar identities for the sequence of Fibonacci polynomials can be shown by standard methods. For example, the following statement is a result of computing the determinants of the matrices in Theorem 7.

Corollary: If $\langle g_n \rangle \in \mathcal{S}_{(a,b)}^{(V)}$ and p is a natural number,

$$g_{p+2}g_p - g_{p+1}^2 = (-b)^{p-1}(g_3 g_1 - g_2^2).$$

For the sequence $W_{(a,b)}$, the identity in the corollary above reduces to

$$w_{p+2}w_p - w_{p+1}^2 = -(-b)^p.$$

If Theorem 7 is specialized to $W_{(a,b)}$ and the result simplified, the following corollary results.

Corollary: For each natural number p,

$$\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} w_{p+2} & w_{p+1} \\ w_{p+1} & w_p \end{pmatrix} = Q^{p+1}.$$

Theorem 8: If p **and** q are natural numbers, $w_{p+q+1} = w_{p+1} \cdot w_{q+1} + bw_p \cdot w_q$.

Proof: When Q^{p+q} is computed directly using the corollary above, w_{p+q+1} is the first row, first column entry. When Q^p and Q^q are computed using the corollary above, and the results multiplied, the first row, first column entry of $Q^p \cdot Q^q$ is $w_{p+1} \cdot w_{q+1} + bw_p \cdot w_q$.

Corollary: If m, n, and j are natural numbers and $n > j$, then

$$w_{m+n+1} = w_{m+j+1}w_{n-j+1} + bw_{m+j}w_{n-j}.$$

This corollary can be proved by simply letting $p = m + j$ and $q = n - j$ in Theorem 8.

Theorem 8 may be used to prove another generalization of itself.

Corollary: If $\{u, v, p\}$ is a set of natural numbers,

$$w_{u+p}w_{v+p} - (-b)^p w_u w_v = w_p w_{u+v+p}.$$

Proof: The proof is by induction on p. If $p = 1$, the corollary reduces to Theorem 8. Suppose that k is a natural number such that $w_{u+k}w_{v+k} - (-b)^k w_u w_v = w_k w_{u+v+k}$.

$$\begin{aligned} w_{u+k+1}w_{v+k+1} - (-b)^{k+1}w_u w_v &= (w_{u+v+2k+1} - bw_{u+k}w_{v+k}) - (-b)^{k+1}w_u w_v \\ &= w_{u+v+2k+1} - b(w_{u+k}w_{v+k} - (-b)^k w_u w_v) \\ &= w_{u+v+2k+1} - bw_k w_{u+v+k} \\ &= w_{u+v+k+1}w_{k+1} + bw_k w_{u+v+k} - bw_k w_{u+v+k} \\ &= w_{k+1}w_{u+v+k+1}. \end{aligned}$$

This corollary can be rearranged to give the following identity, analogous to one previously published for the sequence of Fibonacci numbers [3].

$$w_{u+p}w_{v+p} - w_p w_{p+u+v} = (-b)^p w_u w_v.$$

4. DIVISIBILITY PROPERTIES OF $W_{(a,b)}$

If $b = 0$, $W_{(a,b)} = \langle (ax)^{n-1} \rangle$. If $a = 0$, $W_{(a,b)} = \langle 1, 0, b, 0, b^2, \ldots \rangle$. Divisibility properties for each of these types of sequences are easily studied as separate cases. As a result, throughout the remainder of Section 4, a and b will be assumed to be nonzero numbers.

Theorem 9: If p and q are natural numbers, $w_p | w_{pq}$.

This theorem can be proved by induction, using Theorem 8 and writing

$$w_{p(k+1)} = w_{(kp-1)} + p + 1$$

in the induction step. The converse of Theorem 9 relies on Theorem 9 and a sequence of lemmas.

Lemma 1: If p is a natural number and $p > 1$ and U is a polynomial that divides both w_p and w_{p+1}, then $U | w_{p-1}$.

Proof: Suppose S and T are polynomials and $w_p = U \cdot S$ and $w_{p+1} = U \cdot T$:

$$w_{p-1}(x) = (1/b)(U(x))(T(x) - axS(x)).$$

Lemma 2: If U is a polynomial and there exists a natural number p such that $U | w_p$ and $U | w_{p+1}$, then U has degree 0.

Proof: If $p = 1$, $U | w_1$ and the conclusion follows from the fact that $w_1 = 1$. If $p > 1$, Lemma 1 may be applied repeatedly to show that $U | w_1$.

Lemma 3: If $\{n, p, q, r\}$ is a set of natural numbers and $p > 1$ and $q = np + r$ and $w_p | w_q$, then $w_p | w_r$.

Proof: Since $p > 1$, $np - 1 > 0$, $q = (np - 1) + r + 1$, and so by Theorem 8,

$$w_q = w_{np} \cdot w_{r+1} + bw_{np-1} \cdot w_r.$$

By hypothesis, $w_p | w_q$, and by Theorem 9, $w_p | w_{np}$ and hence $w_p | w_{np}w_{r+1}$. Thus, $w_p | bw_{np-1}w_r$. The greatest common divisor of w_{np} and w_{np-1} is a constant (Lemma 2), and so the greatest common divisor of w_p and w_{np-1} is a constant. Therefore, $w_p | w_r$.

Theorem 10: If p and q are natural numbers and $w_p | w_q$, then $p | q$.

Proof: If $p = 1$, the conclusion is obvious. Suppose $p > 1$. $q > p$, so there exists a pair of nonnegative integers, n and r, such that $q = np + r$ and $0 \le r < p$. $r = 0$ since, if $r > 0$, Lemma 3 establishes that $w_p | w_r$, which is a contradiction, since $r < p$.

REFERENCES

1. Marjorie Bicknell. "A Primer for the Fibonacci Numbers. Part VII: An Introduction to Fibonacci Polynomials and Their Divisibility Properties." *The Fibonacci Quarterly* 8, No. 4 (1970):407-420.
2. M. N. S. Swamy. Problem B-74. *The Fibonacci Quarterly* 3, No. 3 (1965):236.
3. Problem E 1396. *American Math. Monthly* 67 (1960):81-82, 694.

*****

A DIVISIBILITY PROPERTY OF BINOMIAL COEFFICIENTS

CARL S. WEISMAN

University of Rochester, Rochester, New York

Let p be a prime number. Let the integers $a_{n\ell}$ be defined by the identity

$$\binom{py}{n} = \sum_{\ell} a_{n\ell}\binom{y}{\ell}.$$

The purpose of this note is to prove that the exponent to which p divides $a_{n\ell}$ is at least $\ell - (n - \ell)/(p - 1)$.

Let Y be a set with y elements. Let $Y_1, \ldots, Y_p$ be disjoint sets, each equipped with a fixed bijection to Y. We wish to count the subsets N of $Y_1 \cup \cdots \cup Y_p$ having exactly n elements. For such a setset N, denote by N_i the image of $N \cap Y_i$ in Y.

If j is an m-tuple $(i_1, \ldots, i_m)$ with $1 \leq i_1 < i_2 < \cdots < i_m \leq p$, write $i \in \operatorname{supp} j$ if $i = i_k$ for some k.

Let $S_j^m = \{x \in \cup N_i \mid x \in N_i$ if and only if $i \in \operatorname{supp} j\}$. The sets S_j^m are pair-wise disjoint, and $N_i = \cup\{S_j^m \mid i \in \operatorname{supp} j\}$. Moreover, it is easily seen that any change in the ordered p-tuple $(N_1, \ldots, N_p)$ of subsets must change some S_j^m. So producing the sets $N_1, \ldots, N_p$ is the same as producing the sets S_j^m.

Let $L = \cup N_i$, and let ℓ be its cardinality. Let $S^m = \bigcup_j S_j^m$; then S^m consists of the points of L that correspond to exactly m points of N. If t_m is the cardinality of S^m, therefore, one has $n = \ell + \sum_{m=2}^{p} (m - 1)t_m$, and $n/p \leq \ell \leq n$.

We construct as follows. First select a subset L of Y with cardinality ℓ between n/p and n. Then select a subset S^p of L with cardinality t_p at most $(p - 1)^{-1}(n - \ell)$. Then select a subset S^{p-1} of $L - S^p$ with cardinality t_{p-1} at most $(p - 2)^{-1}(n - \ell - (p - 1)t_p)$. Continue in this way until S^3 has been selected as a subset of $L - S^p - \cdots - S^4$ with cardinality t_3 at most $2^{-1}(n - \ell - (p - 1)t_p - \cdots - 3t_4)$. Now select a subset S^2 of $L - S^p - \cdots - S^3$ with cardinality t_2 equal to

$$n - \ell - \sum_{m=3}^{p} (m - 1)t_m.$$

Define $S^1 = L - S^p - \cdots - S^2$ with cardinality t_1. Finally, select a partition of each S^m into $\binom{p}{m}$ subsets S_j^m.

The above procedure yields the following expression for $\binom{py}{n}$:

$$\sum_{\ell}\binom{y}{\ell}\sum_{t_p}\binom{\ell}{t_p}\sum_{t_{p-1}}\binom{\ell - t}{t_{p-1}} \cdots \binom{\ell - t_p - \cdots - t_3}{t_2}\binom{p}{1}^{t_1} \cdots \binom{p}{p-1}^{t_{p-1}},$$

in which the numbers ℓ and t_m are constrained by the equalities and inequalities of the preceding paragraph. In this expression, each term in the coefficient of $\binom{y}{\ell}$ includes a power of p at least $t_1 + \cdots + t_{p-1} = \ell - t_p \geq \ell - (p - 1)^{-1}(n - \ell)$.

*****