(c) $L_{2r}^7 F_n^7 - F_{n+2r}^7 - F_{n-2r}^7 = 7L_{2r}F_{n-2r}F_nF_{n+2r}(F_{n+2r}^2 + L_{2r}F_nF_{n-2r})^2$

(d) $L_{2r}^7 L_n^7 - L_{n+2r}^7 - L_{n-2r}^7 = 7L_{2r}L_{n-2r}L_nL_{n+2r}(L_{n+2r}^2 + L_{2r}L_nL_{n-2r})^2$

The proofs of 4(a) and 4(c) could serve as proof models for the remaining identities.

4(a):
$$F_{n+2r+1}^7 - L_{2r+1}F_n^7 - F_{n-2r-1}^7 = -(L_{2r+1}F_n)^7 + F_{n+2r+1}^7 - F_{n-2r-1}^7$$
$$= -(F_{n+2r+1} - F_{n-2r-1})^7 + F_{n+2r+1}^7 - F_{n-2r-1}^7$$
$$= 7F_{n+2r+1}^6F_{n-2r-1} - 21F_{n+2r+1}^5F_{n-2r+1}^2 + 35F_{n+2r-1}^4F_{n-2r+1}^3 - 35F_{n+2r+1}^3F_{n-2r-1}^4 + 21F_{n+2r+1}^2F_{n-2r-1}^5 - 7F_{n+2r+1}F_{n-2r-1}^6$$
$$= 7F_{n+2r+1}F_{n-2r-1}(F_{n+2r+1}^5 - 3F_{n+2r+1}^4F_{n-2r-1} + 5F_{n+2r+1}^3F_{n-2r-1}^2 - 5F_{n+2r+1}^2F_{n-2r-1}^3 + 3F_{n+2r+1}F_{n-2r-1}^4 - F_{n-2r-1}^5)$$
$$= 7F_{n+2r+1}F_{n-2r-1}(F_{n+2r+1} - F_{n-2r-1})(F_{n+2r+1}^4 - 2F_{n+2r+1}^3F_{n-2r-1} + 3F_{n+2r+1}^2F_{n-2r-1}^2 - 2F_{n+2r+1}F_{n-2r-1}^3 + F_{n-2r-1}^4)$$
$$= 7F_{n+2r+1}F_{n-2r-1}L_{2r+1}F_n(F_{n+2r+1}^2 - F_{n+2r+1}F_{n-2r-1} + F_{n-2r-1}^2)^2$$
$$= 7L_{2r+1}F_{n-2r-1}F_nF_{n+2r+1}(F_{n+2r+1}^2 - L_{2r+1}F_nF_{n-2r-1})^2$$

4(c):
$$F_n^7L_{2r}^7 - F_{n+2r}^7 - F_{n-2r}^7 = (F_nL_{2r})^7 - F_{n+2r}^7 - F_{n-2r}^7 = (F_{n+2r} + F_{n-2r})^7 - F_{n+2r}^7 - F_{n-2r}^7$$
$$= 7F_{n-2r}F_{n+2r}(F_{n+2r}^5 + 3F_{n+2r}^4F_{n-2r} + 5F_{n+2r}^3F_{n-2r}^2 + 5F_{n+2r}^2F_{n-2r}^3 + 3F_{n+2r}F_{n-2r}^4 + F_{n-2r}^5)$$
$$= 7F_{n-2r}F_{n+2r}(F_{n+2r} + F_{n-2r})(F_{n+2r}^4 + 2F_{n+2r}^3F_{n-2r} + 3F_{n+2r}^2F_{n-2r}^2 + 2F_{n+2r}F_{n-2r}^3 + F_{n-2r}^4)$$
$$= 7F_{n-2r}F_{n+2r}L_{2r}F_n(F_{n+2r}^2 + F_{n+2r}F_{n-2r} + F_{n-2r}^2)^2$$
$$= 7L_{2r}F_{n-2r}F_nF_{n+2r}(F_{n+2r}^2 + L_{2r}F_nF_{n-2r})^2$$

NOTE: On the assumption that Type I primitive units are given by
$$\left(\frac{a + b\sqrt{D}}{2}\right)^n = \frac{L_n + F_n\sqrt{D}}{2},$$
these sixteen generalized F-L identities are valid Type I identities.

REFERENCE

1. Problem H-112 (and its solution), proposed by Leonard Carlitz. *The Fibonacci Quarterly* 7 (1969).

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A CHARACTERIZATION OF THE PYTHAGOREAN TRIPLES

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The Pythagorean triples are all the systems of positive integers x, y, z which satisfy the "Pythagorean equation"

(1) $$x^2 + y^2 = z^2.$$

It is well known (see Uspensky and Heaslet [2]) that the Pythagorean triples can be characterized by the formulas

(2) $$x = M(r^2 - s^2),\ y = M2rs,\ z = M(r^2 + s^2),$$

where r and s are any two relatively prime numbers of different parity with $r > s$ and M is an arbitrary positive integer.

In this note we characterize the Pythagorean triples that satisfy (1) in terms of the integer k, where

(3) $$z = y + k$$

for some $k \geq 1$.

The case where $k = 1$ and thus $z = y + 1$ is also well known and a proof appears in Ore [1]. The solutions are characterized by the formulas

(4) $$x = 2n + 1,\ y = 2n(n + 1),\ z = 2n(n + 1) + 1$$

where n is any integer ≥ 1.

In order to generalize the result for all positive integers k, we observe that any positive integer k can be written in the form

(5) $$k = p^2q$$

where p and q are positive integers and $q = P_1P_2 \ldots P_m$ for distinct primes $P_1, P_2, \ldots, P_m$. Consequently, we have the following characterization.

Theorem: Let (x, y, z) be a Pythagorean triple where $z = y + k$ for $k \geq 1$. Then

(i) if k is odd and $k = p^2q$, then for $n \geq 1$,

$$x = pq(2n + p)$$
$$y = 2nq(n + p)$$
$$z = 2nq(n + p) + k,$$

(ii) if k is even and $k = 2p^2q$, then for $n \geq 1$,

$$x = 2pq(n + p)$$
$$y = nq(n + 2p)$$
$$z = nq(n + 2p) + k.$$

Proof: (i) Suppose k is odd, $k = p^2q$ and $q = P_1P_2 \ldots P_m$ where $P_1, P_2, \ldots, P_m$ are distinct odd primes. Then

$$x^2 + y^2 = (y + k)^2$$

implies

$$x^2 = 2yk + k^2$$

or

$$x^2 = p^2(2yq + p^2q^2).$$

Hence,

(6) $$x = p\sqrt{2yq + p^2q^2}.$$

Since x is an integer, $2yq + p^2q^2 = t^2$ for some integer t. Solving for y,

(7) $$y = \frac{t^2 - p^2q^2}{2q}.$$

But y is positive, hence, $t = s + pq$ for some integer $s \geq 1$. Substituting t into (7) yields

(8) $$y = \frac{s(s + 2pq)}{2q}.$$

Hence, s must be even, say $s = 2w$ for some integer $w \geq 1$, and substituting into (8) we have

(9) $$y = \frac{2w(w + pq)}{q}.$$

Since q is odd and a product of distinct primes, q must divide w, i.e., $w = nq$ for some integer $n \geq 1$. Substituting w into (9) yields the desired formula for

(10) $$y = 2nq(n + p),$$

and substituting (10) for y in (6) yields

$$x = pq(2n + p).$$

(ii) Suppose k is even, $k = 2p^2q$ and q is a product of distinct primes. Then

$$x^2 = 4p^2(yq + p^2q^2),$$

and

(11) $$x = 2p\sqrt{yq + p^2q^2}.$$

Again, $yq + p^2q^2 = t^2$ for some integer t. Solving for y,

(12) $$y = \frac{t^2 - p^2q^2}{q}.$$

But y is positive, hence $t = s + pq$ for some integer $s \geq 1$. Substituting t into (12) yields

(13) $$y = \frac{s(s + 2pq)}{q}.$$

Since q is a product of distinct primes, q must divide s, i.e., $s = nq$ for some integer $n \geq 1$. Substituting s into (13) yields the desired formula for y,

(14) $$y = nq(n + 2p),$$

and substituting (14) for y in (11) yields

$$x = 2pq(n + p).$$

REFERENCES

1. O. Ore. *Number Theory and Its History*. New York: McGraw-Hill, 1948.
2. J. V. Uspensky and M. A. Heaslet. *Elementary Number Theory*. New York: McGraw-Hill, 1939.

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ON PRIMITIVE WEIRD NUMBERS

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1. INTRODUCTION

Let n be a positive integer. Denote by $\sigma(n)$ the sum of divisors of n. It is called n perfect if $\sigma(n) = 2n$, abundant if $\sigma(n) \geq 2n$, and deficient if $\sigma(n) < 2n$. Further, n is defined to be pseudoperfect if it is the sum of some of its proper divisors that all are distinct (d is a proper divisor of n, if d/n and $d < n$).

An integer n is called weird if n is abundant but not pseudoperfect. It is primitive abundant if it is abundant but all its proper divisors are deficient. If n is primitive abundant but not pseudoperfect, it is called primitive weird.

It is not known [1] if there are infinitely many primitive weird numbers or any odd weird numbers. A list of weird and primitive weird numbers not exceeding 10^6 is given in [1]. However, there is a misprint in [1] on page 618: instead of 539774 one should read 539744.

In this note we let n specially be of the form

(1) $$n = 2^{\alpha}pq \quad (\alpha \geq 1,\ p < q,\ p \text{ and } q \text{ odd primes}),$$

and give necessary and sufficient conditions under which n is primitive weird. As far as we know this cannot be found in the literature. As an application, we list some primitive weird numbers exceeding 10^6.

Throughout this note, let p and q be odd primes and $p < q$.

We use the following notations:

$$S = \sum_{\nu=0}^{\alpha} 2^{\nu} = 2^{\alpha+1} - 1, \quad S' = \sum_{(\nu)} 2^{\nu}$$

(the sum being taken over some of the indices ν);

$$S_p = \sum_{\nu=0}^{\alpha} 2^{\nu}p = (2^{\alpha+1} - 1)p, \quad S_p^m = S_p - mp \quad (0 \leq m \leq 2^{\alpha+1} - 1);$$

$$S_q = \sum_{\nu=0}^{\alpha} 2^{\nu}q = (2^{\alpha+1} - 1)q, \quad S_q^n = S_q - nq \quad (0 \leq n \leq 2^{\alpha+1} - 1);$$

$$S_{pq} = \sum_{\nu=0}^{\alpha-1} 2^{\nu}pq = (2^{\alpha} - 1)pq, \quad S_{pq}^k = S_{pq} - kpq \quad (0 \leq k \leq 2^{\alpha} - 1).$$