PSEUDO-PERIODIC DIFFERENCE EQUATIONS

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ABSTRACT

Periodic difference equations are generalized to pseudo-periodic (ψ_p) difference equations, and Minkowski's method extended to solve them. This is seen to lead to an identity involving Fibonacci and Lucas sequences.

1. INTRODUCTION

There are no general methods for the solution of all difference equations. However, periodic difference equations having the form

(1.1) $$P(E)f(x) = (a_1, \ldots, a_n)_n$$

where $P(E)$ is some polynomial of the shift operator, $f(x)$ is the unknown function in the discrete variable x and $a_1, \ldots, a_n$ are n constants, can be solved using Minkowski's operational calculus [1]. It would, clearly, be of interest to find a method for solving a more general class of equations.

In this paper we define a wider class of equations,

(1.2) $$P(E)f(x) = (a_1(x), \ldots, a_n(x))_n$$

which are not, strictly speaking, periodic, and call them *pseudo-periodic* (ψ_p) *equations*. We shall be extending Minkowski's method to solve these equations. This will be done using a discrete function, $h(x, m)$, defined by

(1.3) $$h(x, m) = \begin{cases} 1 & \text{when } m/x \\ 0 & \text{when } m\nmid x \end{cases}$$

which has the following properties:

(P1) $[h(x, m)]^j = h(x, m)\ \forall$ integers $j > 0$;

(P2) $\sum_{j=0}^{m-1} h(x + j, m) = 1$;

(P3) $h(x, m_1)h(x, m_2) = h(x, m)$ where $m = (m_1, m_2)$;

(P4) $h(x + mk, m) = h(x, m)\ \forall$ integers k;

(P5) $h(nx, m) = h(x, m)$ where $\langle n, m\rangle = 1$.

We shall then be able to evaluate the expression

(1.4) $$f(x) = \frac{1}{(E^{m_1 m_2} - a^{m_1 m_2})}\left[\frac{nx}{m_1}\right] h(x, m_2)$$

where $\left[\frac{p}{q}\right]$ is the usual bracket function. This will enable us to solve almost all ψ_p difference equations. Clearly we could always use it to solve periodic difference equations.

2. SOLUTION OF ψ_p DIFFERENCE EQUATIONS

We will start by finding the particular solution of the difference equation (Δ being the difference operator $E - 1$):

(2.1) $$\Delta f(x) = \left(\frac{x}{m}\right)^k h(x, m).$$

Using different values of m and k, any polynomial can be constructed by appropriate combinations of terms on the righthand side (apart from functions like $\sqrt{x}$, etc.). Thus, we can construct almost any linear, first-order, ψ_p difference euqation from equation (2.1). This clearly leads to more general ψ_p difference equations.

Consider the difference of the $(k + 1)$st Bernoulli polynomial,

$$\Delta B_{k+1}\left(\left[\frac{x-1}{m}\right]+1\right)=\sum_{i=0}^{k}\binom{k+1}{1}(1+B)^{k-i+1}\left[\frac{x-1}{m}\right]^{i}. \tag{2.2}$$

$B_k \equiv B^k$ being the kth Bernoulli number, using (P2) we can write

$$\Delta B_{k+1}\left(\left[\frac{x-1}{m}\right]+1\right)=\sum_{i=1}^{k+1}(-1)^{k-i+1}\binom{k+1}{1}B_{k-i+1}\sum_{j=1}^{i}(-1)^{j+1}\binom{i}{j}\left(\frac{x}{m}\right)^{i-j}h(x, m).$$

Putting $j = i - r$, changing the order of summation of j and then putting $i = s + r$,

$$\Delta B_{k+1}\left(\left[\frac{x-1}{m}\right]+1\right)=\sum_{r=0}^{k}\sum_{s=1}^{k-r+1}(-1)^{k-r}\binom{k+1}{s+r}\binom{s+r}{r}B_{k-s-r+1}\left(\frac{x}{m}\right)^{r}h(x, m),$$

Now

$$\binom{k+1}{s+r}\binom{s+r}{r}=\binom{k+1}{r}\binom{k-r+1}{s}$$

and

$$\sum_{s=1}^{k-r+1}B_{k-s-r+1}\binom{k-s+1}{s}=(1+B)^{k-r+1}-B^{k-r+1}.$$

$$\therefore\quad \Delta B_{k+1}\left(\left[\frac{x-1}{m}\right]+1\right)=\sum_{r=0}^{k}(-1)^{k-r}\binom{k+1}{r}\left(\frac{x}{m}\right)^{r}[(1-B)^{k-r+1}-B^{k-r+1}]h(x, m)$$

$$=(k+1)\left(\frac{x}{m}\right)^{k}h(x, m).$$

Thus the solution of equation (2.1) is

$$f(x)=\frac{1}{k+1}B_{k+1}\left(\left[\frac{x-1}{m}\right]+1\right)+c. \tag{2.3}$$

Now it can be seen that if

$$\Delta^{-1}f(x)=F(x)+c$$

$$\Delta^{-1}f\left(\left[\frac{x}{m}\right]\right)=\sum_{i=0}^{m-1}F\left(\left[\frac{x+i}{m}\right]\right)+c.$$

Thus, equation (2.3) gives us

$$\Delta^{-1}\left[\frac{x}{m}\right]^{k}=\frac{1}{k+1}\sum_{r=0}^{m-1}B_{k+1}\left[\frac{x+r}{m}\right]$$

$$=\frac{m}{k+1}B_{k+1}\left(\left[\frac{x}{m}\right]\right)-\frac{1}{k+1}\left(x-m\left[\frac{x}{m}\right]\right)\left[\frac{x}{m}\right]^{k}. \tag{2.4}$$

To solve equation (1.4), we write $f(x)$ in the form

$$f(x)=\sum_{i=0}^{k-j}\sum_{j=0}^{k}a_{ij}\left[\frac{nx}{m_1}\right]^{k-i}h(x+j, m), \tag{2.5}$$

where a_{ij} are coefficients chosen to fit the given $f(x)$. Operating on both sides of equation (2.5) with $(E^{m_1m_2} - a^{m_1m_2})$ and comparing with equation (1.3) we see that

$$a_{00}=\frac{1}{1-a^{m_1m_2}}$$

$$a_{i0}=-a_{00}\sum_{s=0}^{i-1}a_{s0}\binom{k-s}{i-s}(m_2n)^{i-s} \tag{2.6}$$

$$a_{ij}=0 \quad \text{for} \quad j\neq 0.$$

Denoting a_{i0} by a_i for all $i = 0, \ldots, k$, we get

$$(2.7) \qquad (E^{m_1 m_2} - a^{m_1 m_2}) \sum_{i=0}^{k} \left[\frac{nx}{m_1}\right]^{k-i} a_i h(x, m_2) = \left[\frac{nx}{m_1}\right]^k h(x, m_2).$$

If we assume that for some j,

$$(2.8) \qquad a_j = \binom{k}{j} (m_2 n)^j \sum_{r=0}^{j} (a_{00})^{r+1} r! S_n^j,$$

S_r^j being Sterling's numbers of the second kind [2]. Then equation (2.6) gives

$$\frac{a_{j+1}}{a_{00}} = -\sum_{q=0}^{j} \binom{k-q}{j+1-q} (m_2 n)^{j+1-q} \binom{k}{q} (m_2 n)^q \sum_{r=0}^{q} (a_{00})^{q+1} S_r^q,$$

since

$$\binom{k-q}{j+1-q}\binom{k}{q} = \binom{k}{j+1}\binom{j+1}{q}$$

and

$$\sum_{q=0}^{j} S_p^q \binom{j+1}{q} = (p+1) S_{p+1}^{j+1}$$

and $S_0^{j+1} = 0$ and $S_p^q = 0$ for $q < p$, we obtain

$$a_{j+1} = \binom{k}{j+1} (m_2 n)^{j+1} \sum_{r=0}^{j+1} (a_{00})^{r+1} r! S_r^{j+1}.$$

As equation (2.8) is true for $j = 0$, it must hold for all values of j. Thus, putting equation (2.8) into equation (2.7), we obtain

$$(2.9) \qquad f(x) = \sum_{i=0}^{k} \sum_{r=0}^{i} \left[\frac{nx}{m_1}\right]^{k-i} \binom{k}{i} \frac{(m_2 n)^i r! S_r^i h(x, m)}{(1 - a^{m_1 m_2})^{r+1}}.$$

Thus, we have a general method for solving the required ψ_p difference equations.

3. AN INTERESTING IDENTITY

We find that by considering a special case of equation (2.7) we are able to obtain an identity between combinations of two well-known sequences—the Fibonacci and Lucas sequences. This is given as an example of the technique given above and its applications. We consider the ψ_p difference equation

$$(3.1) \qquad (E - a)(E - b) f(x) = x^k h(x, m).$$

Writing the particular solution of equation (3.1) as $P_k(x)$, we write

$$P_k(x) = (E - a)^{-1} (E - b)^{-1} x^k h(x, m)$$

$$(3.2) \qquad = \left\{ \sum_{i=1}^{m} \sum_{j=1}^{m} E^{m-i} a^{i-1} E^{m-j} b^{j-1} \right\} \left\{ \sum_{p=0}^{k} \sum_{q=0}^{k} a(k, q) b(k-q, p-q) x^k \right\},$$

where

$$(3.3) \qquad a(k, q) = m^q \binom{k}{q} \sum_{r=0}^{q} \frac{q! S_r^q}{(1 - a^m)^{r+1}}, \qquad b(k, q) = m^q \binom{k}{q} \sum_{r=0}^{q} \frac{q! S_r^q}{(1 - b^m)^{r+1}}$$

Taking $m = 1$ in equation (3.3) and using the requirement that there is symmetry under interchange of a and b, we obtain

$$
\begin{aligned}
a(k, q)b(k - q, p - q) &= \frac{1}{2}\{a(k, q)b(k - q, p - q) + a(k - q, p - q)b(k, q)\} \\
&= \frac{1}{2}\binom{k}{q}\binom{p}{q}\left\{\sum_{i=0}^{M}\sum_{j=1}^{M-i} j!(j + 1)!(-1)^{i+j+1}\ell_i(S_i^q S_{i+j}^{p-q} + S_{i+j}^q S_i^{p-q})_2\right\}
\end{aligned}
\tag{3.4}
$$

where M is the greater of q and $(p - q)$ and ℓ_i is Lucas' sequence, given by

$$\ell_n = \ell_{n-1} + \ell_{n-2} \quad \text{and} \quad \ell_0 = 2,\ \ell_1 = 1.$$

Also, if a and b are roots of the equation $y^2 - y - 1 = 0$, the lefthand side of equation (3.3) should be [3]

$$-\sum_{i=1}^{q} i!F_{i+1}S(q, i)$$

from equation (3.2), F_i being the Fibonacci sequence. Then

$$\sum_{i=0}^{q} i!F_{i+1}S(q, i) = -\frac{1}{2}\binom{p}{q}\sum_{i=0}^{M}\sum_{j=0}^{M-i}(-1)^{i+j+1}j!(j + 1)!\ell_i(S_j^q S_{i+j}^{p-q} + S_{i+j}^q S_j^{p-q}), \tag{3.5}$$

which is the required identity.

CONCLUSION

We have defined ψ_p difference equations as generalizations of the periodic difference equations. This is a much wider class of difference equations than the periodic ones, but does not contain all difference equations. We extended Minkowski's operational calculus to deal with a large class (but not all) ψ_p difference equations. This is of interest in itself as a means of solving more difference equations than Minkowski's calculus enabled us to. It is also of interest inasmuch as it provides an independent means of solving periodic difference equations and thereby discovering new identities between combinations of various sequences. Thus, it can also be regarded as being of interest in number theory.

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SOLUTION OF PSEUDO-PERIODIC DIFFERENCE EQUATIONS

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ABSTRACT

A method for getting the particular solution of pseudo-periodic difference equations, by using a discrete periodic function has been given. Some identities, equalities, and inequalities have been derived by using the above-mentioned discrete function.

1. INTRODUCTION

Periodic difference equations have been previously solved [1] by the use of Minkowski's operational calculus. The type of equations solved by this method are

$$P(E)f(x) = (a_1, a_2, \ldots, a_n)_n, \tag{1.1}$$