120	117	123	122	13	8	12	1	87	82	94	91
121	124	118	119	3	10	6	15	81	92	88	93
114	115	125	128	2	11	7	14	96	85	89	84
127	126	116	113	16	5	9	4	90	95	83	86
47	46	34	35	69	66	79	76	105	98	111	104
33	36	48	45	74	80	65	71	110	99	102	107
40	37	41	44	75	77	68	70	100	109	108	101
42	43	39	38	72	67	78	73	103	112	97	106
52	61	55	58	135	143	138	130	29	17	20	32
54	59	49	64	132	134	139	141	22	26	23	27
57	56	62	51	142	140	133	131	28	24	25	21
63	50	60	53	137	129	136	144	19	31	30	18

FIGURE 5. A 12-by-12 Magic Square

Together the two families contain $8^{17}(110)(8 \cdot 110^8 + 1)$ distinct twelfth-order magic squares.

This technique can be employed to produce two families of knth order magic squares from magic squares of the kth and nth orders. If $k = n$, there is one family. Such is the family of 134,217,728 ninth-order magic squares [2].

REFERENCES

1. William H. Benson & Oswald Jacoby. *New Recreations with Magic Squares*. New York: Dover, 1976. Pp. 187-198.
2. Charles W. Trigg. "A Family of Ninth-Order Magic Squares." *Mathematics Magazine* (accepted for publication).

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COIN TOSSING AND THE r-BONACCI NUMBERS

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In this paper we find the probability that a fair coin tossed n times will contain a run of at least r consecutive heads.

Let $X_n = \{x_1x_2 \ldots x_n / x_i \in \{h,t\},\ i = 1, 2, \ldots, n\}$ be the set of 2^n equi-probable outcomes and Y_n^r be the subset of X_n each of whose elements contains a run of at least r consecutive heads. Also, let $a(r,n)$ be the cardinality of Y_n^r. We can construct Y_n^r by noting that each of its elements must fall into one of the following two categories:

(1) $$HA_{n-r}$$

(2) $$W_j tHA_{n-j-1-r}$$

where H is the first run of r consecutive heads to appear when reading from left to right,

A_i is an i-string of any combination of heads and tails,

W_j is a j-string of heads and tails not containing H, and

t is a singleton tail.

Since there are $2^j - a(r,j)$ ways in which W_j can occur, the total number of elements of type (2) is

$$\sum_{j=0}^{n-1-r} [2^j - a(r,j)]2^{n-j-1-r}.$$

Summing over all possibilities for (1) and (2) we obtain

$$a(r,n) = 2^{n-r} + \sum_{j=0}^{n-1-r} [2^j - a(r,j)]2^{n-j-1-r}$$

$$(3) \qquad = 2^{n-r}\left[1 + (n - r)/2 - \sum_{j=r}^{n-1-r} a(r,j)/2^{j+1}\right].$$

The next three lemmas exhibit some relationships among the $a(r,n)$.

Lemma 1: If $n \geq r$, then

$$a(r,n) = 2^{n-r} + \sum_{j=n-r}^{n-1} a(r,j).$$

Proof: Clearly $a(k,k) = 1$ for all $k > 0$. If we rewrite (3) and assume the lemma true for $n - 1$, we have

$$a(r,n) = 2\left\{2^{n-r-1}\left[1 + (n - 1 - r)/2 - \sum_{j=r}^{n-2-r} a(r,j)/2^{j+1}\right] + 2^{n-2-r} - a(r,n - 1 - r)/2\right\}$$

$$= 2a(r,n - 1) + 2^{n-1-r} - a(r,n - 1 - r)$$

$$= a(r,n - 1) + \left\{2^{n-1-r} + \sum_{j=n-1-r}^{n-2} a(r,j)\right\} + 2^{n-1-r} - a(r,n - 1 - r)$$

$$= 2^{n-r} + \sum_{j=n-r}^{n-1} a(r,j);$$

thus, the lemma holds for n and the proof follows by induction.

The next lemma relates $a(r,n)$ to the r-Bonacci numbers $F_m^{(r)}$ where $F_1^{(r)} = 1$, $F_m^{(r)} = 2^{m-2}$ for $m = 2, \ldots, r + 1$, and $F_m^{(r)} = F_{m-r}^{(r)} + \cdots + F_{m-1}^{(r)}$ for $m = r + 2, r + 3, \ldots$.

Lemma 2: If $r \leq n$, then

$$(4) \qquad a(r,n) = \sum_{j=1}^{n-r+1} F_j^{(r)} 2^{n-r+1-j}.$$

Proof: We know that $a(r,r) = 1 = F_1^{(r)}$, which proves the case $n = r$. For $n > r$, assume that for $i = r, r + 1, \ldots, n - 1$,

$$a(r,i) = \sum_{j=1}^{i-r+1} F_j^{(r)} 2^{i-r+1-j}$$

then

$$a(r,n) = 2^{n-r} + \sum_{i=n-r}^{n-1} a(r,i)$$

$$= 2^{n-r} + \sum_{i=n-r}^{n-1}\left\{\sum_{j=1}^{i-r+1} F_j^{(r)} 2^{i-r+1-j}\right\}$$

$$= 2^{n-r} + \sum_{j=1}^{n-r}\left\{\sum_{i=j-r+1}^{j} F_i^{(r)}\right\}2^{n-r-j}$$

$$= 2^{n-r} + \sum_{j=1}^{n-r}\left\{\sum_{i=1}^{r} F_{j-r+1}^{(r)}\right\}2^{n-r-j},$$

which reduces to (4) and the proof is completed by induction.

The last lemma, proved by Swamy [4], is a generalization of a problem posed by Carlitz [3].

Lemma 3: $\sum_{j=1}^{m} F_j^{(r)} 2^{m-j} = 2^{m+r+1} - F_{m+r+1}^{(r)}.$

We are now in a position to calculate the desired probability.

Theorem: The probability that n tosses of a fair coin will contain a run of at least r consecutive heads, $r \leq n$, is given by $1 - F_{n+2}^{(r)}/2^n$.

Proof: Apply Lemma 3 to (4) with $m = n - r + 1$.

REFERENCES

1. S. Bezuszka and L. D'Angelo. "An Application of Tribonacci Numbers." *The Fibonacci Quarterly* 15 (1977):140-144.
2. M. Bicknell and V. E. Hoggatt, Jr. "Generalized Fibonacci Polynomials." *The Fibonacci Quarterly* 11 (1973):457-465.
3. L. Carlitz. Problem B-135. *The Fibonacci Quarterly* 6 (1968):90.
4. M. N. S. Swamy. "A Formula for $\sum_{k=1}^{n} F_k(x)y^{n-k}$ and Its Generalization to r-Bonacci Polynomials." *The Fibonacci Quarterly* 15 (1977):73-77.

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COMBINATORIAL IDENTITIES DERIVED FROM UNITS

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ABSTRACT

We shall derive two combinatorial identities by considering units in infinite classes of cubic fields. This is a comparatively new application of units.

0. INTRODUCTION

We shall begin by stating a result of Bernstein and Hasse [2] concerning systems of units in infinitely many fields.

Theorem: Let $P(x)$ be a polynomial of degree $n \geq 2$ with the form

$$P(x) = (x - D_0)(x - D_1) \ldots (x - D_{n-1}) - d,\ d \geq 1,\ D_i, d \in Z,\ D_0 \equiv D_i \pmod{d},$$
$$D_0 - D_i \geq 2d(n-1),\ (i = 1, \ldots, n-1),\ D_0 > D_1 > \ldots > D_{n-1}.$$

Then $P(x)$ has exactly n distinct real roots; $P(x)$ is irreducible over Q; and if w is the largest root of $P(x)$, then

$$e_i = \frac{(w - D_i)^n}{d} \qquad (i = 0, \ldots, n-1)$$

are different units of $Q(w)$. Furthermore, any $n - 1$ of these units form a system of independent units.

1. COMBINATORIAL IDENTITIES FROM UNITS

Consider the cubic polynomials $P(x) = (x - D_0)(x - D_1)(x - D_2) - 1$; D_i as above. First we work with the case $D_2 = 0$; later we will eliminate this condition. Now it is clear that itself is a unit in $Q(w)$ with $N(w) = 1$. We proceed by expressing the integral powers of w. For any integer $n \geq 0$, let

$$w^n = x_n + y_n w + z_n w^2 \qquad (x_n, y_n, z_n \in Z). \tag{1.1}$$

Calculating directly and taking into account that $w^3 = 1 - Bw + Aw^2$ where $A = D_0 + D_1$ and $B = D_0 D_1$, we have

$$\begin{aligned} w^{n+1} &= z_n + (x_n - Bz_n)w + (y_n + Az_n)w^2 \\ w^{n+2} &= (y_n + Ax_n) + (z_n - By_n - ABz_n)w + (x_n - Bz_n + Ay_n + A^2 z_n)w^2; \end{aligned} \tag{1.2}$$

so that

$$x_{n+1} = z_n;\ y_{n+1} = x_n - Bx_{n+1};\ z_{n+1} = x_{n-1} - Bx_n + Ax_{n+1}. \tag{1.3}$$

From (1.1) and (1.3), we obtain

$$w^n = x_n + (x_{n-1} - Bx_n)w + (x_{n-2} - Bx_{n-1} + Ax_n)w^2 \tag{1.4}$$