THE PENTANACCI NUMBERS

PAUL N. MENDELSOHN
350 E. 87th St., New York, NY 10028

The elegance of the Fibonacci sequence lies in the fact that its simple definition gives rise to a multitude of properties. Similar qualities can be found in a Pentanacci sequence defined as:

$$P_{(0)} = 0$$
$$P_{(n)} = \text{any five chosen integers, } n = 1, 2, 3, 4, 5$$
$$P_{(n)} = \sum_{m=n-5}^{n-1} P_{(m)}, \; n > 5$$

The generalized form of a Pentanacci sequence is, therefore:

$$p, q, r, s, t, (p + q + r + s + t), (p + 2q + 2r + 2s + 2t), (2p + 3q + 4r + 4s + 4t), \ldots$$

We will consider the specific Pentanacci sequence in which $p = q = r = s = t = 1$. This series begins 1, 1, 1, 1, 1, 5, 9, 17, 33, 65, 129, 253, ...

There is a simple recursive function for finding the sum of any n consecutive Pentanacci numbers. By definition,

$$\sum_{n=1}^{N} P_{(n)} > P_{(N+1)}, \; N > 5,$$

since

$$\begin{aligned} P_{(1)} + P_{(2)} + P_{(3)} + \cdots + P_{(N-4)} + P_{(N-3)} + P_{(N-2)} + P_{(N-1)} + P_{(N)} \\ > P_{(N-4)} + P_{(N-3)} + P_{(N-2)} + P_{(N-1)} + P_{(N)} \\ = P_{(N+1)} \end{aligned}$$

When we subtract $P_{(N-4)} + P_{(N-3)} + P_{(N-2)} + P_{(N-1)} + P_{(N)}$ from both sides, we arrive at $P_{(1)} + P_{(2)} + P_{(3)} + \cdots + P_{(N-5)} > 0$. This immediately leads to:

$$\sum_{n=1}^{N} P_{(n)} = P_{(N+1)} + \sum_{n=1}^{N-5} P_{(n)}, \; N > 5.$$

In general,

$$\begin{aligned} \sum_{k=M}^{N} P_{(k)} &= \sum_{k=1}^{N} P_{(k)} - \sum_{k=1}^{M-1} P_{(k)} \\ &= P_{(N+1)} + \sum_{k=1}^{N-5} P_{(k)} - P_{(M)} - \sum_{k=1}^{M-6} P_{(k)}. \end{aligned}$$

THE PENTANACCI RATIOS AND THEIR DEFINING FIFTH-POWER EQUATIONS

It is well known that the ratio of two consecutive Fibonacci numbers, $F_{(n+1)}/F_{(n)}$, approaches the limit $\frac{1 + \sqrt{5}}{2} = 1.618034$ and its reciprocal approaches $\frac{1 - \sqrt{5}}{2} = 0.618034$. These limits are the roots of $X^2 - X - 1 = 0$. The ratio of two consecutive Pentanacci numbers, $P_{(n+1)}/P_{(n)}$, approaches the limit 1.9659482 and its reciprocal approaches 0.5086604. These ratios are the only real roots of the fifth-power equation $X^5 - X^4 - X^3 - X^2 - X - 1 = 0$.

By definition, $P_{(n+1)} = P_{(n)} + P_{(n-1)} + P_{(n-2)} + P_{(n-3)} + P_{(n-4)}$. Dividing through by $P_{(n-1)}$, we define:

$$P_{(n)}/P_{(n-1)} = Z_1$$
$$P_{(n-1)}/P_{(n-3)} = Z_2 = Z_1^2 = P_{(n+1)}/P_{(n-1)}$$
$$P_{(n-1)}/P_{(n-4)} = Z_3 = Z_1^3$$

This gives us $Z_1^2 - Z_1 + 1 + 1/Z_1 + 1/Z_1^2 + 1/Z_1^3$, from which the quintic equation, $Z^5 - Z^4 - Z^3 - Z^2 - Z - 1 = 0$, is derived.

CONTINUED FRACTION EXPANSION OF PENTANACCI RATIOS

The ratios $P_{(n+1)}/P_{(n)}$ and $P_{(n)}/P_{(n+1)}$ can be expressed as finite continued fractions in order to demonstrate that they are rational numbers. In general, a continued fraction may be represented as:

$$[a_1, a_2, a_3, \ldots] \quad \text{or} \quad a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\cdots}}}}$$

The terms, a_i, are known as partial quotients. A finite continued fraction has a finite number of partial quotients and represents a rational number. Infinite continued fractions have an infinite number of partial quotients and represent irrational numbers.

It can be seen that:

$$\text{(A)} \qquad P_{(n+1)}/P_{(n)} = [1, a_2, a_3, a_4, \ldots, a_n]$$

$$\text{(B)} \qquad P_{(n)}/P_{(n+1)} = [0, 1, a_3, a_4, \ldots, a_{n+1}]$$

In equation (b), $a_{(k+1)}$ is the same as the a_k of equation (a) for all k, $1 \leq k \leq n$.

Consider a/b, where $a > b$ and both a and b are integers.

$$a/b = c + a/b - c,$$

$$a/b = c + (a - cb)/b$$

or

$$\text{(C)} \qquad a/b = c + \cfrac{1}{\cfrac{b}{a - cb}}$$

This can be expanded further.

Now consider b/a, where $a > b$ and both a and b are integers.

$$b/a = 0 + b/a,$$

so

$$\text{(D)} \qquad b/a = 0 + \cfrac{1}{\cfrac{a}{b}}$$

Applying equation (C) to equation (D) gives rise to

$$\text{(E)} \qquad b/a = 0 + \cfrac{1}{c + \cfrac{1}{\cfrac{b}{a - cb}}}$$

This also can be expanded by further manipulation of the $b/(a - cb)$ term.

THE GOLDEN RECTANGLE AND INTERMEDIATE SEQUENCES

Another property of the Fibonacci sequence is that two consecutive Fibonacci numbers represent the lengths of the sides of the Golden Rectangle. A Golden Rectangle is shown in Figure 1. Segments creating smaller Golden Rectangles and a square are included in the figure. The lengths of the sides of all the quadrangles are Fibonacci numbers.

A similar Pentanacci rectangle is shown in Figure 2. Note from Figure 2 that $a = P_{(n)} - P_{(n-1)}$, $b = P_{(n-1)} - P_{(n-2)}$, $c = P_{(n-2)} - P_{(n-3)}$, $d = P_{(n-3)} - P_{(n-4)}$ and $e = P_{(n-4)} - P_{(n-5)}$. In the Fibonacci sequence $F_{(n)} - F_{(n-1)} = F_{(n-2)}$. In the Pentanacci sequence, however, $P_{(n)} - P_{(n-1)} \neq P_{(n-2)}$. By subtracting two consecutive Pentanacci numbers, a new sequence called an Intermediate Sequence is formed.

The first few members of the Pentanacci sequence and of the first two intermediate sequences are:

1, 1, 1, 1, 1, 5, 9, 17, 33, 65, 129, 253, ...
0, 0, 0, 0, 4, 4, 8, 16, 32, 64, 124, ...
0, 0, 0, 4, 0, 4, 8, 16, 32, 60, ...

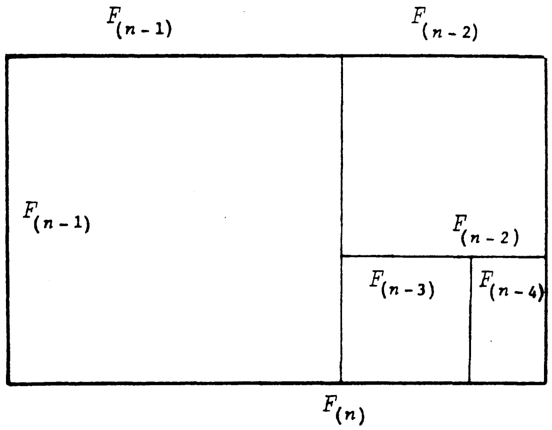

FIGURE 1

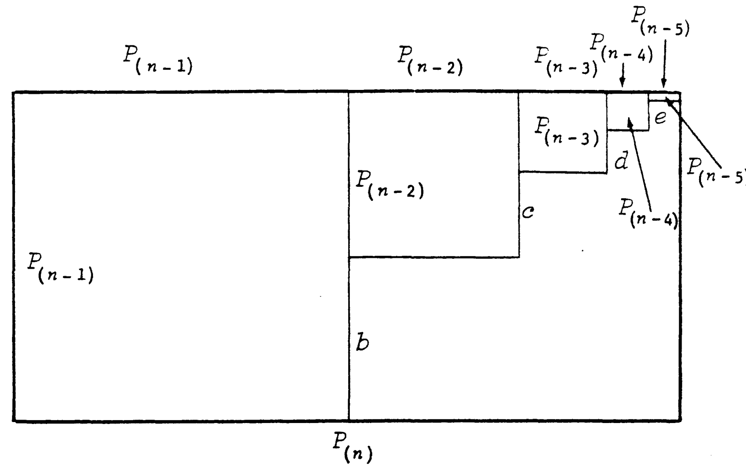

FIGURE 2

It can be shown that each intermediate sequence is a Pentanacci sequence. From the definition of the Pentanacci numbers:

$$P_{(k+1)} = P_{(k)} + P_{(k-1)} + P_{(k-2)} + P_{(k-3)} + P_{(k-4)}$$
$$P_{(k)} = P_{(k-1)} + P_{(k-2)} + P_{(k-3)} + P_{(k-4)} + P_{(k-5)}.$$

So, $P_{(k+1)} - P_{(k)} = P_{(k)} - P_{(k-5)}$ for $k \geq 5$, and for $k < 5$.

Given the following equations:

$$\begin{aligned}
P_{(k+1)} - P_{(k)} &= P_{(k)} - P_{(k-5)} \\
P_{(k)} - P_{(k-1)} &= P_{(k-1)} - P_{(k-6)} \\
P_{(k-1)} - P_{(k-2)} &= P_{(k-2)} - P_{(k-7)} \\
P_{(k-2)} - P_{(k-3)} &= P_{(k-3)} - P_{(k-8)} \\
P_{(k-3)} - P_{(k-4)} &= P_{(k-4)} - P_{(k-9)}
\end{aligned}$$

The sum of the right-hand side terms is $\sum_{n=k-4}^{k} P_{(n)} - \sum_{m=k-9}^{k-5} P_{(m)}$ which is equal to $P_{(k+1)} - P_{(k-4)}$, the sequence member following $P_{(k)} - P_{(k-5)}$ as defined by the definitions of both the Pentanacci sequence and an intermediate sequence.

The sum of the right-hand side terms, $P_{(k+1)} - P_{(k-4)}$, also equals $P_{(k+2)} - P_{(k+1)}$, the difference between the next two members of the Pentanacci sequence. Hence, we have shown, by applying the definitions of the Pentanacci and intermediate sequences that the latter is a subset of the former.

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