we would obtain the same identities with A and B replaced by $\overline{A} = -2D_2 + D_0 + D_1$ and $\overline{B} = (D_2 - D_0)(D_2 - D_1)$, respectively.

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A STOLARSKY ARRAY OF WYTHOFF PAIRS

DAVID R. MORRISON

Harvard University, Cambridge, MA 02138

A *positive Fibonacci sequence* is a sequence $\{s_k\}$ such that $s_{k+1} = s_k + s_{k-1}$ and $s_k > 0$ for k sufficiently large. A *Stolarsky array* is an array $A = \{A_{m,n}:m,n \quad 0\}$ of natural numbers such that:

(a) the rows $\{A_{m,1}, A_{m,2}, \ldots\}$ are positive Fibonacci sequences;
(b) every natural number occurs exactly once in the array;
(c) every positive Fibonacci sequence is a row of the array, after a shift of indices. That is, given a positive Fibonacci sequence $\{s_j\}$, there exist m and k such that

$$A_{m,n} = s_{n+k}.$$

The first such array[1] was constructed by Stolarsky [8]. In this note, we will construct a new Stolarsky array using Wythoff pairs. By inspecting the tables of these two arrays, it is easy to obtain more Stolarsky arrays. (For example, in either table, the 4 may be shifted from the second to the third row.) It would be interesting to have a classification of the Stolarsky arrays.

Let $\alpha = \frac{1}{2}(1 + \sqrt{5})$ be the golden ratio, and let $[\ ]$ denote the greatest integer function. The *Wythoff pairs* are the pairs of numbers $([n\alpha], [n\alpha^2])$ which give the winning positions in Wythoff's game (see [5], for example). These pairs have two remarkable properties:

1. *Beatty complementarity* [2]—Every natural number m is either of the form $[n\alpha]$ or of the form $[n\alpha^2]$, but not both.

2. *Connell's formula* [4]—

$$[n\alpha] + [n\alpha^2] = [[n\alpha^2]\alpha].$$

<u>*Lemma 1*</u>: Let $s_1 = [k\alpha]$, $s_2 = [k\alpha^2]$ generate a positive Fibonacci sequence. Then (s_{2j-1}, s_{2j}) is a Wythoff pair for every $j > 0$.

<u>*Proof*</u>: Since $\alpha^2 = \alpha + 1$, we have

$$(*) \qquad n + [n\alpha] = [n\alpha^2].$$

Suppose $(s_{2j-3}, s_{2j-2}) = ([m\alpha], [m\alpha^2])$ is a Wythoff pair. Then by Connell's formula,

$$s_{2j-1} = [m\alpha] + [m\alpha^2] = [[m\alpha^2]\alpha],$$

while by formula (*),

$$s_{2j} = [m\alpha^2] + [[m\alpha^2]\alpha] = [[m\alpha^2]\alpha^2].$$

Thus, (s_{2j-1}, s_{2j}) is a Wythoff pair, and the lemma follows by induction.

We define the *Wythoff array* to be an array $W = \{W_{m,n}\}$ which is Fibonacci in its rows, and is generated by:

$$W_{m,1} = [[m\alpha]\alpha],\ W_{m,2} = [[m\alpha]\alpha^2].$$

The first 100 terms of the Wythoff array are listed in Table 1.

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[1]That (c) holds for Stolarsky's array does not seem to have been noticed. We will verify it as Corollary 2, below.

TABLE 1

1	2	3	5	8	13	21	34	55	89
4	7	11	18	29	47	76	123	199	322
6	10	16	26	42	68	110	178	288	466
9	15	24	39	63	102	165	267	432	699
12	20	32	52	84	136	220	356	576	932
14	23	37	60	97	157	254	411	665	1076
17	28	45	73	118	191	309	500	809	1309
19	31	50	81	131	212	343	555	898	1453
22	36	58	94	152	246	398	644	1042	1686
25	41	66	107	173	280	453	733	1186	1919

According to Lemma 1, $(W_{m,2k-1}, W_{m,2k})$ is a Wythoff pair for every m, k.

Lemma 2: The Wythoff array contains all the Wythoff pairs.

Proof: First note that $(W_{1,1}, W_{1,2}) = ([\alpha], [\alpha^2])$. Suppose the $\{[n\alpha], [n\alpha^2]\} \subset W$ for all $n < N$. By Beatty complementarity, $N = [m\alpha]$ or $N = [m\alpha^2]$. If $N = [m\alpha^2]$, then by Lemma 1, one of the rows of W contains

$$\ldots, [m\alpha], [m\alpha^2], [N\alpha], [N\alpha^2], \ldots .$$

On the other hand, if $N = [m\alpha]$, then $(W_{m,1}, W_{m,2}) = ([N\alpha], [N\alpha^2])$. In either case, $\{[N\alpha], [N\alpha^2]\} \subset W$, and the lemma is proved by induction.

Corollary: The Wythoff array contains each natural number exactly once.

After a shift of indices, any positive Fibonacci sequence $\{s_j\}$ will satisfy $0 \le 2s_0 < s_1$ (see [1]). We say that such a sequence is in *standard form*.

Theorem: Let $\{s_j\}$ be a Fibonacci sequence in standard form. For any $\omega > 0$, and for all sufficiently large k,

$$s_{2k} < s_{2k-1}\alpha < s_{2k} + \omega.$$

Proof: A simple induction shows that

$$s_j = s_0 F_{j+2} + (s_1 - 2s_0)F_j ,$$

where $\{F_j\}$ is Fibonacci's sequence $F_0 = F_1 = 1$. Using the continued fraction expansion for α (cf. [7]) and the formula

$$F_{2k+1}F_{2k-1} = F_{2k}^2 + 1,$$

we see that

$$\frac{F_{2k}}{F_{2k+1}} < \alpha < \frac{F_{2k-1}}{F_{2k}} = \frac{F_{2k}}{F_{2k-1}} + \frac{1}{F_{2k}F_{2k-1}},$$

so that

$$F_{2k} < F_{2k-1}\alpha < F_{2k} + \frac{1}{F_{2k}}.$$

Let k be large enough so that

$$\frac{s_0}{F_{2k+2}} + \frac{s_1 - 2s_0}{F_{2k}} < \omega.$$

Then

$$\begin{aligned} s_{2k} &= s_0F_{2k+2} + (s_1 - 2s_0)F_{2k} \\ &< s_0F_{2k+1}\alpha + (s_1 - 2s_0)F_{2k-1}\alpha \\ &= s_{2k-1}\alpha \\ &< s_0F_{2k+2} + (s_1 - 2s_0)F_{2k} + \frac{s_0}{F_{2k+2}} + \frac{s_1 - 2s_0}{F_{2k}} \\ &< s_{2k} + \omega, \end{aligned}$$

which proves the theorem.

Corollary 1: The Wythoff array is a Stolarsky array.

Proof: Let $\{s_j\}$ be a Fibonacci sequence in standard form. We must show that $\{s_j\}$ is a row of W, after a shift of indices. For k large, we have

$$s_{2k} < s_{2k-1}\alpha < s_{2k} + 1,$$

so that $[s_{2k-1}\alpha] = s_{2k}$. Thus, by formula (*), (s_{2k}, s_{2k+1}) is a Wythoff pair. The corollary now follows from Lemma 2.

Corollary 2: Stolarsky's array is a Stolarsky array.

Proof: We recall the definition of Stolarsky's array. Let $g(x) = [x\alpha + \frac{1}{2}]$. It is easy to check that for any natural number k, $\{k, g(k), g^2(k), \ldots\}$ forms a positive Fibonacci sequence. Stolarsky's array $S = \{S_{m,n}\}$ is defined by

$$S_{1,1} = 1;$$

$$S_{m,1} = \text{the smallest natural number not in } \{S_{k,n} : k < m\};$$

$$S_{m,n} = g^{n-1}(S_{m,1}).$$

The first 100 terms of Stolarsky's array are listed in Table 2.

TABLE 2

1	2	3	5	8	13	21	34	56	89
4	6	10	16	26	42	68	110	178	288
7	11	18	29	47	76	123	199	322	521
9	15	24	39	63	102	165	267	432	699
12	19	31	50	81	131	212	343	555	898
14	23	37	60	97	157	254	411	665	1076
17	28	45	73	118	191	309	500	809	1309
20	32	52	84	136	220	356	576	932	1508
22	36	58	94	152	246	398	644	1042	1686
25	40	65	105	170	275	445	720	1165	1885

By construction, S satisfies condition (a). Condition (b) was proved by Hendy [6; Theorem 1]. To check condition (c), we apply the above theorem with $\omega = \frac{1}{2}$. For k large enough,

$$s_{2k} < s_{2k-1}\alpha < s_{2k-1}\alpha + \tfrac{1}{2} < s_{2k} + 1,$$

so that $g(s_{2k-1}) = [s_{2k-1}\alpha + \frac{1}{2}] = s_{2k}$. By Hendy's theorem, $s_{2k-1} = S_{m,n}$ for some m, n. But then, $s_{2k} = g(S_{m,n}) = S_{m,n+1}$, so that $\{s_j\}$ is the mth row of S, after a shift in indices.

Stolarsky's conjecture, proved by Butcher [3] and by Hendy [6], says that

$$\{S_{m,2} - S_{m,1}\} = \{S_{m,1}\} \cup \{S_{m,2}\}.$$

There is an analogous statement for the Wythoff array:

Proposition: $\{W_{m,2} - W_{m,1}\} = \bigcup_{k \geq 0} \{W_{m,2k+1}\}$.

Proof: $W_{m,2} - W_{m,1} = [[m\alpha]\alpha^2] - [[m\alpha]\alpha] = [m\alpha]$ by formula (*). Since $(W_{m,2k+1}, W_{m,2k+2})$ is always a Wythoff pair,

$$\{[m\alpha]\} = \bigcup_{k \geq 0} \{W_{m,2k+1}\}.$$

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