TRIANGULAR ARRAYS ASSOCIATED WITH SOME PARTITIONS

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A partition of a positive integer n by a set of integers S is set $S' = \{s_i\}$ $(1 \le i \le r)$ of integers s_i drawn from S such that

$$n = \sum_{i=1}^{r} s_i . \tag{1}$$

The general problem for pattitions is to discuss the number of partitions of n by S, a number that depends, in each particular problem, on the restrictions that are placed on the representation (1); for example, r may not be fixed, S' may or may not contain repetitions, or the order of the terms in (1) may or may not matter according to the question at issue. A typical, but difficult, problem is that of unrestricted partitions: what is the number $p(n)$ of partitions of n by S when S is the set of all positive integers, repetitions are allowed, and neither the number of terms nor their order in (1) matters? (See [2, pp. 273-296] for an introduction to the theory of partitions as well as an account of some results concerning $p(n)$.)

A much simpler problem for which the answer is known is: what is the number $b(n,k)$ of partitions of $(n + 1)$ by S when S is the set of all positive integers, $r = k + 1$, repetitions are allowed, and the order of the terms in (1) matters? The number in question, as may easily be seen, is just a binomial coefficient

$$b(n,k) = \binom{n}{k}, \quad n \ge k \ge 0. \tag{2}$$

The associated triangular array $\{b(n,k)\}$ $(n \ge k \ge 0)$ is the familiar Pascal triangle and, among many identities for the $b(n,k)$, we have

$$b(n,k) = \sum_{m=1}^{n-k+1} b(n - m,\ k - 1), \quad n \ge k \ge 1. \tag{3}$$

Introducing the generating functions

$$B_k(x) = \sum_{n=k}^{\infty} b(n,k)x^{n-k}; \; B(x) = B_0(x) = \sum_{n=0}^{\infty} x^n,$$

we have, at least formally,

$$B_k(x) = [B(x)]^{k+1}, \quad k \ge 1 \tag{4}$$

$$(1 - x)B(x) = 1. \tag{5}$$

Hale [1] has recently enquired about partitions using k ones rather than partitions into k parts: what is the number $f(n,k)$ of partitions of n by S when S is the set of all integers, r is arbitrary, repetitions are allowed, the order of terms matters, and $s_i = 1$ for exactly k values of i, $1 \le i \le r$? Carson and Oates, in reply [3], have given a formula analogous to (3), namely,

$$f(n,k) = f(n - 1,\ k - 1) + \sum_{m=2}^{n-k} f(n - m,\ k), \quad n \ge k + 2 \ge 3, \tag{6}$$

noting also that, for $n \ge 2$, the $f(n,\ 0)$ are the Fibonacci numbers, since

$$f(n,\ 0) = f(n - 1,\ 0) + f(n - 2,\ 0), \quad n \ge 3, \tag{7}$$

with

$$f(1,\ 0) = 0; \; f(2,\ 0) = 1,$$

Since, as Hale [1] noted,

$$f(n,n) = 1; \; f(n,\ n - 1) = 0, \quad n \ge 1,$$

the associated triangular array $\{f(n,k)\}$ $(n \ge k \ge 0;\ n \ge 1)$ is completely determined by (6, 7).

To prove (7), note that either the first (and only) term s_1 in (1) is n or form some m, $2 \le m < n$, $s_1 = m$, and then the remaining terms $\{s_i\}$ $(2 \le i \le r)$ form a partition of $(n - m)$ by S. Hence

$$(8) \qquad f(n,\ 0) = 1 + \sum_{m=2}^{n-1} f(n - m,\ 0), \qquad n \geq 3$$

$$= 1 + \sum_{m=2}^{n-2} f(n - 1 - m,\ 0) + f(n - 2,\ 0), \qquad n \geq 4$$

$$= f(n - 1,\ 0) + f(n - 2,\ 0), \qquad n \geq 4$$

where the last equation also holds for $n = 3$. The proof of (6) is similar to that of (8), except that now $s_1 = 1$ is possible, giving the additional term $f(n - 1,\ k - 1)$. Notice that taking

$$(9) \qquad f(0,\ 0) = 1$$

both gives an apex to the triangular array $\{f(n,k)\}$ and eases the above proofs, allowing (7) to be extended to $n = 2$. Moreover, taking $f(n,k) = 0$ for $k < 0$ or $k > n$, allows (6) to be extended to $n \geq k + 2 \geq 2$ with (8) as a special case.

An alternative approach, complementing this additive theory, is by way of convolutive or multiplicative identities analogous to (4). By way of illustration, consider $f(n,\ 1)$, so that exactly one of the s_i in (1) is equal to 1. Now either $s_1 = 1$ and $\{s_i\}$ $(2 \leq i \leq r)$ is a partition of $(n - 1)$ by S, or for some i, $1 < i < r$, $s_i = 1$, and then for some m, $2 \leq m \leq n - 1$, $\{s_j\}$ $(1 \leq j < i)$ is a partition of m by S, while $\{s_j\}$ $(1 < i \leq r)$ is a partition of $(n - m - 1)$ by S; or $s_r = 1$ and $\{s_i\}$ $(1 \leq i < r)$ is a partition of $(n - 1)$ by S. Since these cases are exclusive and exhaustive,

$$f(n,\ 1) = f(n - 1,\ 0) + \sum_{m=2}^{n-1} f(m,\ 0)\ (n - m - 1,\ 0) + f(n - 1,\ 0), \quad n \geq 3$$

or, making use of (9),

$$(10) \qquad f(n,\ 1) = \sum_{m=1}^{n} f(n,\ 0) f(n - m - 1,\ 0), \quad n \geq 2.$$

Similarly, by considering the least i $(1 \leq i \leq r)$ for which $s_i = 1$ in (1), we have for $k \geq 1$

$$f(n,k) = f(n - 1,\ k - 1) + f(1,\ 0) f(n - 2,\ k - 1) + \cdots + f(n - k,\ 0) f(k - 1,\ k - 1), \qquad n \geq k + 1$$

$$(11) \qquad = \sum_{m=0}^{n-k} f(m,\ 0) f(n - m - 1,\ k - 1), \qquad n \geq k \geq 1$$

On introducing the generating functions

$$F_k(x) = \sum_{n=k}^{\infty} f(n,k) x^{n-k};\ F(x) = F_0(x),$$

we have, from (10, 11),

$$F_1(x) = F_0(x) F_0(x) = [F(x)]^2$$

$$F_k(x) = F_{k-1}(x) F_0(x) = F_{k-1}(x) F(x), \quad k \geq 1,$$

and it follows that

$$(12) \qquad F_k(x) = [F(x)]^{k+1}, \qquad k \geq 1.$$

From (12), which is the analogue of (4), further identities may be obtained in turn, for example,

$$F_k(x) = F_s(x) F_{k-s-1}(x), \quad 0 \leq s < k.$$

Moreover, by (7, 9), $F(x)$ satisfies the functional equation [cf. (5)],

$$(13) \qquad (1 - x - x^2) F(x) = 1 - x.$$

Similar results hold if we now take S to be the set of the first ℓ positive integers $(\ell \geq 2)$ rather than the set of all integers. Thus, let $b_\ell(n,k)$ be the number of partitions of $n + 1$ by S_ℓ where $S_\ell = \{i\}$ $(1 \leq i \leq \ell;\ \ell \geq 2)$, $r = k + 1$, repetitions are allowed in (1) and the order of the terms in (1) matters; and let $f_\ell(n,k)$ be the number of partitions of n by S_ℓ, r is arbitrary, repetitions are allowed, order matters, and k of the s_i are equal to 1. We further make the conventions that

$$b_\ell(n,k) = 0 = f\ (n,k), \quad k < 0 \text{ or } k > n,$$

$$f_\ell(0,0) = 1,$$

then the results for the triangular arrays $\{b_\ell(n,k)\}$, $\{f_\ell(n,k)\}$ $(n \geq k \geq 0)$ are truncated versions of those for the case of unrestricted S and may be summarized as follows, the proofs also being similar to those above.

First, we have the additive recurrence relations [cf. (3, 6)],

(14a) $$b_\ell(n,k) = \sum_{m=1}^{\ell} b_\ell(n-m,\ k-1), \qquad n \geq k \geq 1,$$

(14b) $$b_\ell(n,0) = 1, \qquad 0 \leq n < \ell; = 0, \qquad n \geq \ell,$$

and

(15a) $$f_\ell(n,k) = f_\ell(n-1,\ k-1) + \sum_{m=2}^{\ell} f_\ell(n-m,\ \ell), \qquad n \geq k \geq 0,$$

(15b) $$f_\ell(0,0) = 1;\ f_\ell(1,0) = 0.$$

Secondly, writing

$$B_{k,\ell}(x) = \sum_{n=k}^{\infty} b_\ell(n,k)x^{n-k};\ F_{k,\ell}(x) = \sum_{n=k}^{\infty} f_\ell(n,k)x^{n-k},$$

$$B_\ell(x) = B_{0,\ell}(x);\ F_\ell(x) = F_{0,\ell}(x),$$

we have [cf. (4, 12)]

(16) $$B_{k,\ell}(x) = [B_\ell(x)]^{k+1};\ F_{k,\ell}(x) = [F_\ell(x)]^{k+1}, \quad k \geq 1,$$

with, from (14b, 15, $k = 0$)

(17) $$(1-x)B_\ell(x) = 1 + x^\ell;\ \left(1 - \sum_{2}^{\ell} x^m\right)F_\ell(x) = 1.$$

Moreover, since

$$b_\ell(n,k) = b(n,k), \qquad n - \ell < k \leq n,$$
$$f_\ell(n,k) = f(n,k), \qquad n - \ell \leq k \leq n,$$

we have, in a natural way (see [2, p. 275]) the limiting results

(18) $$\lim_{\ell\to\infty} B_{k,\ell}(x) = B_k(x);\ \lim_{\ell\to\infty} F_{k,\ell}(x) = F_k(x).$$

Not all partition problems have the multiplicative structure exhibited in (4, 12, 16). For example, returning to the problem of unrestricted partitions mentioned at the beginning, let $p(n,k)$ be the number of partitions of n by S when S is the set of all positive integers, repetitions are allowed, neither the number of terms nor their order in (1) matters, and k of the s_i in (1) are equal to 1, and let

$$P_k(x) = \sum_{n=k}^{\infty} p(n,k)x^{n-k};\ P(x) = P_0(x).$$

Then, without determining $P(x)$, it is at least straightforward to show that

(19) $$P_k(x) = P(x).$$

Shapiro [4] has asked whether there is an arithmetic of triangular arrays where a simple function of the generating function of the first column yields the generating function of the other columns as in (4, 12, 16) and indeed also (19). For example, given a sequence $\{a_n\}$ $(n \geq 0)$, we may define a triangular array $\{t_{n,k}\}$ by

(20a) $$t_{n,k} = \sum_{m=0}^{n-k} a_n t_{n-1,\,k-1+m},$$

(20b) $$t_{0,0} = a_0,$$

and

(20c) $$t_{n,k} = 0, \quad k < 0 \text{ or } k > n.$$

and then if

(21a) $$T_k(x) = \sum_{n=k}^{\infty} t_{n,k} x^{n-k};\ T(x) = T_0(x),$$

(21b) $$T_k(x) = [T(x)]^{k+1}.$$

Conversely, given a triangular array satisfying (21), we may recover a sequence $\{a_n\}$ $(n \geq 0)$ via (20). What are the sequences arising in this way in the partition problems considered above [see (4, 12, 16)]?

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BREAK-UP OF INTEGERS AND BRACKET FUNCTIONS IN TERMS OF BRACKET FUNCTIONS

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ABSTRACT

We have presented a general formula for the break-up of integers into bracket functins, and some formulas for the break-up of bracket functions into other bracket functions.

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It is interesting to find break-ups of variable integers into a sum of bracket functions involving the integer we want to break up and other integers. Two well-known examples of this are

(1) $$x = \sum_{i=0}^{m-1} \left[\frac{x+i}{m}\right] \quad \text{integers } m > 0;$$

(2) $$x = \left[\frac{(p+1)x}{2p+1}\right] + \sum_{i=1}^{p} \left[\frac{x+2i}{2p+1}\right] \quad \text{integers } p > 0.$$

Here we shall find a general break-up of the variable integer into bracket functions involving two other integers (equation 12). The above-mentioned break-ups are special cases of this more general formula.

To derive the general formula, we shall need to use the h-function (defined in [1]) defined by

(3) $$\begin{cases} h(x, m) = 1 & \text{if } m/x \\ = 0 & \text{if } m\not| x \end{cases}$$

It is easily seen that it satisfies the following properties (which we shall use later);

(4) $$\{h(x, m)\}^j = h(x, m) \quad \text{integers } j > 0;$$

(5) $$\sum_{j=1}^{m} h(x+j, m) = 1;$$

(6) $$h(x, m_1)h(x, m_2) = h(x, m) \text{ where } m = (m_1, m_2);$$

(7) $$h(x + mk, m) = h(x, m) \quad \text{integers } k;$$

(8) $$h(nx, m) = h(x, m) \quad \text{if } \langle n, m\rangle = 1.$$

Now, considering the difference operator, Δ, acting on the bracket function $\left[\frac{x-1}{m}\right]$: