TABLE 1

$i_3 - i_4$ (mod 4)	k (mod 5)
0	4
1	2
2	3
3	1

The next case, when 7 divides n, is naturally a bit more complicated. Defining i_2, i_3, i_4, i_5, i_6 in the same manner as before, we obtain

$$k(1^{i_2})(2^{i_3})(3^{i_4})(4^{i_5})(5^{i_6}) \equiv -1 \pmod{7}.$$

Inverse pairs are 2, 4 (of order 3) and 3, 5 (of order 6, so we have

Theorem 4: $k(2^{i_3-i_5})(3^{i_4-i_6}) \equiv -1 \pmod 7$, where $i_3 - i_5$ may be reduced modulo 3 and $i_4 - i_6$ may be reduced modulo 6.

This relationship is shown in Table 2.

TABLE 2

$i_3 - i_5$ (mod 3) \ $i_4 - i_6$ (mod 6); k (mod 7)	0	1	2	3	4	5
0	6	2	3	1	5	4
1	3	1	5	4	6	2
2	5	4	6	2	3	1

The same method can be applied to whatever case is desired: the next case, when 11 divides n, yields a four-dimensional table with 2500 entries.

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ON FIBONACCI NUMBERS OF THE FORM $x^2 + 1$

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Let F_n (n nonnegative) be the nth term of the Fibonacci sequence, defined by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$, and let L_n (n nonnegative) be the nth term of the Lucas sequence, defined by $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$. In a previous paper [3], we showed that the equation

(1) $$F_n = y^2 + 1$$

holds only for $n = 1, 2, 3$, and 5. However, the proof given was quite complicated and depended upon some deep properties of units in quartic fields. Recently, Williams [4] has given a simpler solution of (1) which depends on some very pretty identities involving the Fibonacci and Lucas numbers. In this note, we present a completely elementary solution of

(1) which uses neither algebraic number theory nor the identities employed by Williams. In the course of our investigation we shall use the following theorems, which we state without proof.

Theorem 1: L_n and F_n satisfy the relation

(2) $$L_n^2 - 5F_n^2 = 4(-1)^n.$$

Also, if $x^2 - 5y^2 = 4$, then $x = L_{2n}$ and $y = F_{2n}$ for some n.

Theorem 2 [1]:

(a) If $L_n = x^2$, then $n = 1$ or 3.
(b) If $L_n = 2x^2$, then $n = 0$ or 6.
(c) If $F_n = x^2$, then $n = 0, 1, 2,$ or 12.
(d) If $F_n = 2x^2$, then $n = 0, 3,$ or 6.

Theorem 3 [2]: The only nonnegative integer solutions of the equation $x^2 - 5y^4 = 4$ are $(x, y) = (2, 0), (3, 1)$, and $(322, 12)$.

We now return to our problem and first prove

Lemma 1: If $F_m = 3y^2$, then $y = 0$ or 1.

Proof: If m is odd, Theorem 1 yields $L_m^2 - 45y^4 = -4$, which is impossible mod 3. If m is even and not divisible by 3, then $F_{2m} = F_m L_m$ and $(F_m, L_m) = 1$. Thus, either $F_m = u^2$, $L_m = 3v^2$, or $F_m = 3u^2$, $L_m = v^2$. By Theorem 2 the first case holds only for $m = 1$ or 2 since $3 \nmid m$. If $m = 1$, we get $L_m = 1 \neq 3v^2$. If $m = 2$, we get $F_m = 1$, $L_m = 3$, $y = 1$. Finally, $L_m = v^2$ implies $m = 1$ (by Theorem 2) but then $F_1 \neq 3u^2$.

Next, suppose m is even and $3 \mid m$. Then $(F_m, L_m) = 2$ and $F_{2m} = F_m L_m$ implies $F_m = 2u^2$, $L_m = 6v^2$, or $F_m = 6u^2$, $L_m = 2v^2$. By Theorem 2, the first case only holds for $m = 0, 3,$ or 6, but then $L_m \neq 6v^2$. Finally, the second case implies $m = 0$ or 6, but $F_m = 6v^2$ only for $m = 0$.

Corollary 2: The only nonnegative integer solutions of the equation $x^2 - 45y^4 = 4$ are $(x, y) = (2, 0)$ and $(7, 1)$.

Proof: The equation $x^2 - 45y^4 = 4$ implies $F_{2m} = 3y^2$ for some m. By Lemma 1, $y = 0$ and $y = 1$ are the only possible solutions.

Now we return to (1). If n is odd (1) and (2) yield

(3) $$5x^4 + 10x^2 + 1 = y^2,$$

where x and y are nonnegative integers.

Here $(x, y) = 1$ and we may write (3) as

(4) $$\frac{(5x^2 + 1 + y)}{2}\frac{(5x^2 + 1 - y)}{2} = 5x^4.$$

Any common factor of the two expressions on the left-hand side of (4) divides y and $5x^4$ and, thus, since $5 \nmid y$ these two factors are relatively prime. Thus, we conclude that

$$\frac{5x^2 + 1 + y}{2} = u^4, \quad \frac{5x^2 + 1 - y}{2} = 5v^4, \text{ i.e., } u^4 + 5v^4 = 5x^2 + 1, \ uv = x.$$

This yields $u^4 + 5v^4 - 5u^2v^2 = 1$, which may be written $(2u^2 - 5v^2)^2 - 5v^4 = 4$. By Theorem 3, the only nonnegative integers v satisfying this equation are $v = 0, 1,$ or 12. The solution $v = 12$ yields $u^2 = 521$, which is impossible. The solution $v = 0$ yields $u = 1$. Finally, $v = 1$ yields $u = 1$ and $u = 2$. Thus, if n is odd (1) holds only for $n = 1, 3,$ and 5.

Next, suppose n is even. Then (1) and (2) yield

(5) $$5x^4 + 10x^2 + 9 = y^2.$$

To solve (5) we note, first of all, that $3 \mid y$. Next, we prove that $3 \mid x$. To see this, we compute F_n mod 16 and find that (1) can hold only when $n \equiv 2$ or 8 (mod 24). But if $n \equiv 8$ (mod 24), then $4 \mid n$, $3 \mid F_n$ and F_n cannot be of the form $x^2 + 1$. If $n \equiv 2$ (mod 24), then $F_n \equiv 1$ (mod 3) and if $F_n = x^2 + 1$, then $3 \mid x$. Thus, (5) reduces to

(6) $$45X^4 + 10X^2 + 1 = Y^2.$$

Equation (6) may be written

$$\frac{(Y - 5X^2 - 1)}{2}\frac{(Y + 5X^2 + 1)}{2} = 5X^4,$$

and it is easily shown that the two factors on the left-hand side of this equation are relatively prime. Thus, we conclude

$$\frac{Y + 5X^2 + 1}{2} = u^4, \quad \frac{Y - 5X^2 - 1}{2} = 5v^4,$$

which yields

$$u^4 - 5v^4 = 5X^2 + 1, \; uv = X, \text{ i.e., } u^4 - 5u^2v^2 - 5v^4 = 1.$$

This equation may be written $(2u^2 - 5v^2)^2 - 45v^4 = 4$. By Corollary 2, this equation holds only for $v = 0$ and 1, but only $v = 0$ yields a solution, namely $u = 1$. Thus, the only solution of (4) is $x = 3$, $y = 0$. So if n is even and $F_n = x^2 + 1$, then $n = 2$.

In conclusion, we note that Williams [4] has shown that the complete solution of (1) implies that the only integer solutions of the equation $(x - y)^7 = x^5 - y^5$ with $x > y$ are (1, 0) and (0, -1).

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FIBONACCI SEQUENCE CAN SERVE PHYSICIANS AND BIOLOGISTS

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PART I: SOME OF THEIR SPECIAL NEEDS

All of earth's living beings can react suitably to a range of circumstances. They react to certain stimuli. How much they react is related to how much is the stimulus. Biologists have found that over a wide range of intensity, a proportionate change in stimulus calls forth the same change in response.

Two examples of this relationship are:

1. the nervous system of an animal recognizes as increase in stimulus the same proportional change across most of the range of stimulus;
2. the immune system of an animal responds to the same proportional change in challenge across most of a very wide range.

From (1) derives the Weber-Fechner law of sensory perception. It is a generalization from a wealth of data. Thus, a certain person feels as heavier as little as 11 ounces compared to 10 ounces, and 11 grams compared to 10 grams, and 11 pounds compared to 10 pounds, as well as 110 pounds compared to 100 pounds. Across a range of five-thousandfold, that person distinguishes the same proportionate difference of 10 percent. Note that the basic distinction is not 1 gram nor 1 pound nor 10 pounds, but remains 10 percent.

From (1) likewise derives that some person can hear one musical note as sharper or flatter than another when it is as little as 0.5 percent sharper or flatter, whether the note is tested at the basso's CC, or the coloratura's ccc, which is five octaves higher with soundwaves vibrating 32 times faster than CC. Indeed, the person's range of perception of musical notes may extend beyond two-hundredfold, much more than 32.

With example (2), the immune system, the range often stretches beyond a millionfold. Throughout that range, the amount of offending protein (antigen) that elicits a given rise in the body's immune substance (antibody) stays proportional to the amount of antibody already present.

So in the workaday world of (1) a biologist measuring changes of taste-sensitivity, or of (2) a physician treating a patient's allergic disorder by regular periodic injections of a solution of an offending protein allergen, either scientist should seek to lay out a schedule of ever-increasing strength of solutions, the increase usually being at a constant proportional rate.