Theorem: Let $\{w_n\}$ be an integral linear recurrence of order $m \geq 2$. If no ratio of distinct roots of $P_w(x)$ is a root of unity, then $\{w_n\}$ has infinitely many distinct prime divisors.

Here is how to prove it. Consider the recurrence equation for w_n. Following Gel'fond [1] (or other books on difference equations), the general solution can be expressed as

$$w_n = \frac{1}{D}(P_1(x)\alpha_1^x + \cdots + P_r(x)\alpha_r^x)$$

where $\alpha_1, \ldots, \alpha_r$ are the roots of $P_w(x)$, $P_i(x)$ a polynomial of degree equal to the multiplicity of w_i minus 1 and with algebraic integer coefficients, and D a nonzero determinant of algebraic integers (hence an algebraic integer as well). It easily follows that the conditions for Pólya's theorem are satisfied and $\{w_n\}$ must have infinitely many distinct prime divisors.

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WHAT A DIFFERENCE A DIFFERENCE MAKES!

JERRY T. SULLIVAN

Two men are leaving the office when one remarks that both his wife and boy are celebrating their birthdays that night. The other wonders if it is his youngest son. "Yes," says the first, "but he's not so little anymore. His age, multiplied by my wife's age, is equal to the square of the difference of their ages plus one year." This problem, similar to an earlier one in *The Fibonacci Quarterly* [1], provides some surprising and amusing mathematical twists.

On the premise that many mothers are between 25 and 35 years of age, and also that a typical boy is about 10 years old, pairs of ages such as 10 and 30, 11 and 35, etc., can be tested. After a few trials, an answer is seen to be 13 and 34. Further thought shows that the problem can be handled algebraically. If the age of the wife is W and that of the boy is B, then

(1) $$WB = (W - B)^2 + 1.$$

The wife's age can be solved as a function of the boy's age:

(2) $$W = [3B \pm (5B^2 - 4)^{1/2}]/2.$$

Substituting $B = 13$ into equation (2) and using the positive square root gives the known answer $W = 34$. However, using the negative square root gives the answer $W = 5$. It is an unusual wife who is younger than her son, but the numbers 13 and 5 also satisfy equation (1). Using the number 5 in equation (2) and choosing the negative root gives the numbers 5 and 2 as another solution. Proceeding in this fashion results in the sequence

(3) $$1, 2, 5, 13, 34, 89, \ldots,$$

where each successive pair of numbers satisfies equation (1). The number 1 has the unusual property of giving the solutions 1 and 2 when substituted into equation (2). It does not give a solution lower than itself.

The above sequence is every other number of the usual Fibonacci sequence. Calling the initial age in the sequence A_0, the next A_1, etc., equation (1) may be rewritten as a difference equation,

(4) $$A_{N+1}A_N = (A_{N+1} - A_N)^2 + 1.$$

Equation (4) is a nonlinear difference euqtion, which fortunately can be simplified. First rewrite equation (4) as

$$A_{N+1}^2 - 3A_{N+1}A_N + A_N^2 = -1.$$

This must also hold for the next number pair so that

$$A_{N+2}^2 - 3A_{N+2}A_{N+1} + A_{N+1}^2 = A_{N+1}^2 - 3A_{N+1}A_N + A_N^2.$$

Cancelling like terms and rearranging gives

$$[A_{N+2} - A_N](A_{N+2} - 3A_{N+1} + A_N) = 0.$$

Setting the term in brackets equal to 0 would result in a repeating solution to equation (4), which would not generate the correct age sequence. The correct simplification of equation (4) is the linear difference equation

$$A_{N+2} - 3A_{N+1} + A_N = 0. \qquad (5)$$

Equation (5) is solved by assuming that $A_N = R^N$. Substitution results in

$$R^N(R^2 - 3R + 1) = 0.$$

There are two roots which satisfy the quadratic equation,

$$R_+ = (3 + \sqrt{5})/2 \quad \text{and} \quad R_- = 1/R_+.$$

The solution to the difference equation (5) is

$$A_N = aR^N + bR^{-N},$$

and choosing the constants a and b so that $A_0 = 1$ and $A_1 = 2$ finally results in

$$A_N = (R/R + 1)R^N + (1/R + 1)R^{-N}. \qquad (6)$$

The curious property that $A_0 = 1$ seemed to be a natural boundary for the problem, and is mirrored in the solution. Suppose there were lower solutions A_{-1}, A_{-2}, etc. Replacing N by $-N$ in equation (6) leads to

$$A_{-N} = \left(\frac{R}{R+1}\right)R^{-N} + \left(\frac{1}{R+1}\right)R^N = \left(\frac{R}{R+1}\right)R^{N-1} + \left(\frac{1}{R+1}\right)R^{-(N-1)} = A_{N-1},$$

so that all of the supposed lower solutions are actually equal to a higher one. Lastly, to actually compute A_N from equation (6) is not as formidable as it first appears. It is not necessary to compute large integer powers of $R = (3 + \sqrt{5})/2$, but merely to use the rules

$$R^2 = 3R - 1$$

$$R^3 = R(R^2) = 3R^2 - R = 8R - 3$$

$$R^4 = R(R^3) = 8R^2 - R = 21R - 8$$

$$\vdots$$

etc.

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