By substituting $\lambda = L_{2k}/2$ in the above matrix R, we obtain

$$R_{2k} = \frac{1}{F_{2k}}\begin{pmatrix} F_{(2n+2)k} & F_{2nk} \\ -F_{2nk} & -F_{(2n-2)k} \end{pmatrix}.$$

Also, substituting $\lambda = L_{2k}/2$ into $u_{n+1}(\lambda) = 2\lambda u_n(\lambda) - u_{n-1}(\lambda)$ yields the expression for the general element in the upper left of R_{2k}^n as given in equation (25).

Since we could also show that

$$f_{n+1}(L_{2k+1}) = \sum_{j=0}^{[n/2]} \binom{n-j}{j} L_{2k+1}^{n-2j}$$

by substituting $\lambda = L_{2k+1}$ into the recursion formula for the Fibonacci polynomials, and since also $f_{n+1}(L_{2k+1}) = F_{(2k+1)(n+1)}/F_{2k+1}$, we can generalize equation (25) to the following:

$$F_{(n+1)p}/F_p = \sum_{j=0}^{[n/2]} (-1)^{j(p+1)}\binom{n-j}{j} L_p^{n-2j}, \quad p \neq 0,$$

which was a problem posed by H. H. Fern [4].

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ANTIMAGIC PENTAGRAMS WITH LINE SUMS IN ARITHMETIC PROGRESSION, $\Delta = 3$

CHARLES W. TRIGG

2404 Loring Street, San Diego, CA 92109

A pentagram or five-pointed star can be formed by extending the sides of a regular pentagon until they meet. This figure consists of five equal line segments that form a closed path. Each line intersects every other line, so that there are four intersections or vertices on each line, and two lines at each vertex.

A magic pentagram is formed by distributing ten elements on the vertices of a pentagram in such a way that the sum of the four elements (quartet) on one line equals each of the other four line sums. It has been shown [1, 2, 3, 4, 5] that no magic pentagram can be formed with the first ten positive integers.

An antimagic pentagram is one with five different line sums. Those formable with the first ten positive integers are formidably numerous. We restrict our search to those with five line sums in arithmetic progression and a common difference, $\Delta = 3$. In the sum of the five line sums, each element appears twice, so $5[2a + 4(3)]/2 = 2(55)$. Hence, the progression must be 16, 19, 22, 25, and 28.

The partitions of the five terms of this progression into four elements each < 11 are exhibited in Table 1. To make the table compact, 10 is recorded as X. Designate any quartet with a sum of x as an x-quartet. For the purposes of this discussion, two integers are said to be complementary if their sum is 11. Two quartets are complementary and two pentagrams are complementary if their corresponding elements are complementary.

To construct an antimagic pentagram, we start with the 16-quartet ($\underline{1}$, $\underline{2}$, $\underline{3}$, X) and seek a 19-quartet with which it has exactly one element in common, such as ($\underline{3}$, $\underline{7}$, 4, $\underline{5}$). A 22-quartet with exactly one element in common with each of these is ($\underline{2}$, $\underline{5}$, 6, $\underline{9}$). A 25-quartet with exactly one element in common with each of these three quartets is ($\underline{1}$, $\underline{7}$, 8, $\underline{9}$). The unduplicated elements, which are not underscored, in these four quartets form the 28-quartet (4, 6, 8, X). These five quartets can be distributed on the vertices of a pentagram with their line sums intact, as in Figure 1. Proceeding in this fashion to exhaust Table 1, we find 94 distributions exist in complementary pairs as, for example, in Figures 1 and 2.

TABLE 1

PARTITIONS OF LINE SUMS, $\Delta = 3$

16	19	22	25	28
1 2 3 X	1 2 6 X	1 2 9 X	1 5 9 X	1 8 9 X
1 2 4 9	1 2 7 9	1 3 8 X	1 6 8 X	2 7 9 X
1 2 5 8	1 3 5 X	1 4 7 X	1 7 8 9	3 6 9 X
1 2 6 7	1 3 6 9	1 4 8 9	2 4 9 X	3 7 8 X
1 3 4 8	1 3 7 8	1 5 6 X	2 5 8 X	4 5 9 X
1 3 5 7	1 4 5 9	1 5 7 9	2 6 7 X	4 6 8 X
1 4 5 6	1 4 6 8	1 6 7 8	2 6 8 9	4 7 8 9
2 3 4 7	1 5 6 7	2 3 7 X	3 4 8 X	5 6 7 X
2 3 5 6	2 3 4 X	2 3 8 9	3 5 7 X	5 6 8 9
	2 3 5 9	2 4 6 X	3 5 8 9	
	2 3 6 8	2 4 7 9	3 6 7 9	
	2 4 5 8	2 5 6 9	4 5 6 X	
	2 4 6 7	2 5 7 8	4 5 7 9	
	3 4 5 7	3 4 5 X	4 6 7 8	
		3 4 6 9		
		3 4 7 8		
		3 5 6 8		
		4 5 6 7		

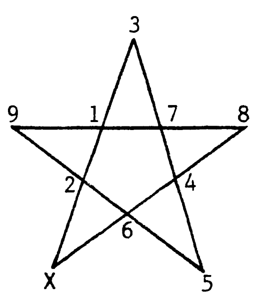

FIGURE 1

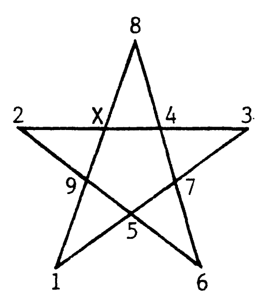

FIGURE 2

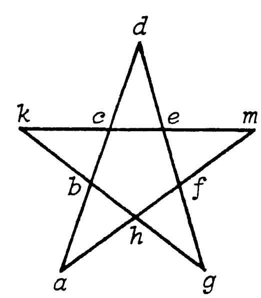

FIGURE 3

One of each complementary pair is listed in Table 2. To facilitate ready construction of any antimagic pentagram from its tabular entry, the vertices have been lettered continuously as in Figure 3. The 16-quartet (a, b, c, d) and the 19-quartet (d, e, f, g) are immediately evident in the table, while the 22-quartet (g, h, b, k), the 25-quartet (k, c, e, m), and the 28-quartet (m, f, h, a) are easily identified. The pentagrams are listed in the order of the appearance of the 16-quartets in Table 1. The asterisks (*) designate the distributions wherein the consecutive digits 2, 3, 4, 5, 6 appear in some order on the vertices of the constituent pentagon.

The distribution of the elements as recorded in Table 2 was made so that in progressing clockwise about the five-line closed path of the pentagram, the line sums would be in increasing order of magnitude. "Essentially, a particular element can appear only in one of two positions—a starpoint or a pentagon vertex. For any quartet, if one element is positioned, the other members can be permuted into 3! orders. Thus any quartet can appear on its line in exactly 2(3!) or 12 orders, not counting reflections. It follows from the tightly interwoven relationship of the quartets that every basic pattern on the pentagram can appear in 12 different guises, all having the same five quartets" [6].

The family of 12 antimagic pentagrams to which the first pentagram in Table 2 belongs is given in Table 3, with 2 in the restricted positions. The 16-, 19-, 22-, 25-, and 28-quartets may be represented by A, B, C, D, and E, respectively. The clockwise and counterclockwise orders of the quartets along each pentagram's closed path are shown in Table 3. The orders in the family comprise all the cyclic permutations of the five quartets.

TABLE 2

ANTIMAGIC PENTAGRAMS WITH LINE SUMS IN A. P., Δ = 3

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*k*	*m*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*k*	*m*
X	2	1	3	7	4	5	6	9	8	8	2	5	1	6	9	3	7	X	4
*X	2	3	1	5	6	7	4	9	8	8	2	5	1	9	3	6	X	4	7
*X	2	3	1	6	4	8	5	7	9	8	2	5	1	9	6	3	X	7	4
X	2	3	1	8	6	4	7	9	5	8	5	2	1	6	9	3	4	X	7
X	2	3	1	9	4	5	8	7	6	8	5	2	1	9	3	6	7	4	X
X	3	1	2	7	4	6	5	8	9	6	1	2	7	4	5	3	8	X	9
X	3	1	2	8	5	4	6	9	7	6	2	1	7	5	4	3	8	9	X
*X	3	2	1	6	5	7	4	8	9	6	2	7	1	5	4	9	8	3	X
X	3	2	1	9	5	4	7	8	6	6	2	7	1	5	X	3	8	9	4
4	2	9	1	3	X	5	8	7	6	6	7	2	1	X	5	3	8	4	9
4	2	9	1	5	6	7	X	3	8	7	1	6	2	5	3	9	8	4	X
9	1	2	4	5	3	7	6	8	X	7	1	6	2	5	9	3	8	X	4
9	1	2	4	7	5	3	8	X	6	7	6	1	2	5	8	4	3	9	X
*9	4	2	1	5	6	7	3	8	X	7	6	1	2	X	4	3	8	5	9
9	4	2	1	7	8	3	5	X	6	8	1	3	4	7	6	2	9	X	5
9	4	2	1	8	3	7	6	5	X	8	4	3	1	7	9	2	6	X	5
9	4	2	1	X	5	3	8	7	6	8	4	3	1	X	6	2	9	7	5
5	2	8	1	3	6	9	7	4	X	5	1	3	7	4	6	2	9	X	8
5	2	8	1	6	9	3	X	7	4	5	7	3	1	X	6	2	9	4	8
5	8	2	1	6	9	3	4	7	X	7	1	5	3	6	8	2	9	X	4
5	8	2	1	9	6	3	7	4	X	7	5	1	3	X	4	2	9	6	8
*8	2	5	1	3	6	9	4	7	X	4	1	5	6	3	8	2	9	X	7
8	2	5	1	3	9	6	4	X	7	4	1	6	5	2	9	3	8	X	7
8	2	5	1	6	3	9	7	4	X										

TABLE 3

ANTIMAGIC PENTAGRAM FAMILY WITH COMMON LINE ELEMENTS

										Sequences of Sums	
a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*k*	*m*	Clockwise	Counterclockwise
X	2	1	3	7	4	5	6	9	8	*ABCDE*	*AEDCB*
X	2	3	1	7	8	9	6	5	4	*ADCBE*	*AEBCD*
3	2	X	1	8	7	9	5	6	4	*ADCEB*	*ABECD*
3	2	1	X	8	4	6	5	9	7	*AECDB*	*ABDCE*
1	2	3	X	4	8	6	9	5	7	*AECBD*	*ADBCE*
1	2	X	3	4	7	5	9	6	8	*ABCED*	*ADECB*
2	3	X	1	8	9	7	5	4	6	*ADBEC*	*ACEBD*
2	3	1	X	8	6	4	5	7	9	*AEBDC*	*ACDBE*
2	X	3	1	7	9	8	6	4	5	*ADEBC*	*ACBED*
2	X	1	3	7	5	4	6	8	9	*ABEDC*	*ACDEB*
2	1	3	X	4	6	8	9	7	5	*AEDBC*	*ACBDE*
2	1	X	3	4	5	7	9	8	6	*ABDEC*	*ACEDB*

It is not customary to count rotations and reflections of configurations as separate arrangements. With this qualification, there are 2(47)(12) or 1128 distinct antimagic pentagrams with line sums forming an arithmetic progression that has a common difference of 3.

There are other antimagic pentagrams with line sums in arithmetic progressiong having common differences of 1 [7], 2 [8], and 4 [9].

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TWO FAMILIES OF TWELFTH-ORDER MAGIC SQUARES

CHARLES W. TRIGG
2404 Loring Street, San Diego, CA 92109

A family of 24,769,797,950,537,728 twelfth-order magic squares can be generated from the basic 9-digit third-order magic square (1) of Figure 1 and the 880 basic fourth-order magic squares.

First, add 9 to each element of square (1) to form square (2) in Figure 1, and repeat the operation until the fifteen derived squares of Figure 1 have been formed. Each of these squares is magic and remains magic in eight orientations: the square itself, its rotations through 90°, 180°, and 270°, and the mirror images of these four.

	(1)			*(2)*			*(3)*			*(4)*	
8	1	6	17	10	15	26	19	24	35	28	33
3	5	7	12	14	16	21	23	25	30	32	34
4	9	2	13	18	11	22	27	20	31	36	29
	(5)			*(6)*			*(7)*			*(8)*	
44	37	42	53	46	51	62	55	60	71	64	69
39	41	43	48	50	52	57	59	61	66	68	70
40	45	38	49	54	47	58	63	56	62	72	65
	(9)			*(10)*			*(11)*			*(12)*	
80	73	78	89	82	87	98	91	96	107	100	105
75	77	79	84	86	88	93	95	97	102	104	106
76	81	74	85	90	83	94	99	92	103	108	101
	(13)			*(14)*			*(15)*			*(16)*	
116	109	114	125	118	123	134	127	132	143	136	141
111	113	115	120	122	124	129	131	133	138	140	142
112	117	110	121	126	119	130	135	128	139	144	137

FIGURE 1. Sixteen 3-by-3 Magic Squares

To construct twelfth-order magic squares, divide a 12-by-12 grid into sixteen 3-by-3 grids, thus forming a 4-by-4 grid of grids. Label this 4-by-4 grid with the elements of one of the basic fourth-order magic squares, such as the familiar pandiagonal square: