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TWO FAMILIES OF TWELFTH-ORDER MAGIC SQUARES

CHARLES W. TRIGG
2404 Loring Street, San Diego, CA 92109

A family of 24,769,797,950,537,728 twelfth-order magic squares can be generated from the basic 9-digit third-order magic square (1) of Figure 1 and the 880 basic fourth-order magic squares.

First, add 9 to each element of square (1) to form square (2) in Figure 1, and repeat the operation until the fifteen derived squares of Figure 1 have been formed. Each of these squares is magic and remains magic in eight orientations: the square itself, its rotations through 90°, 180°, and 270°, and the mirror images of these four.

	(1)			*(2)*			*(3)*			*(4)*	
8	1	6	17	10	15	26	19	24	35	28	33
3	5	7	12	14	16	21	23	25	30	32	34
4	9	2	13	18	11	22	27	20	31	36	29
	(5)			*(6)*			*(7)*			*(8)*	
44	37	42	53	46	51	62	55	60	71	64	69
39	41	43	48	50	52	57	59	61	66	68	70
40	45	38	49	54	47	58	63	56	62	72	65
	(9)			*(10)*			*(11)*			*(12)*	
80	73	78	89	82	87	98	91	96	107	100	105
75	77	79	84	86	88	93	95	97	102	104	106
76	81	74	85	90	83	94	99	92	103	108	101
	(13)			*(14)*			*(15)*			*(16)*	
116	109	114	125	118	123	134	127	132	143	136	141
111	113	115	120	122	124	129	131	133	138	140	142
112	117	110	121	126	119	130	135	128	139	144	137

FIGURE 1. Sixteen 3-by-3 Magic Squares

To construct twelfth-order magic squares, divide a 12-by-12 grid into sixteen 3-by-3 grids, thus forming a 4-by-4 grid of grids. Label this 4-by-4 grid with the elements of one of the basic fourth-order magic squares, such as the familiar pandiagonal square:

1	14	4	15
8	11	5	10
13	2	16	3
12	7	9	6

In each small 3-by-3 grid place the 3-by-3 square, in any of its eight orientations, that has the same identification number as the grid. In forming the twelfth-order square in Figure 2, a different orientation has been given to each of the 3-by-3 squares in the first two rows of the 4-by-4 grid. The same procedure has been followed in filling th last two rows. Indeed, the orientations are such that the square in Figure 2 is pandiagonal. (In a pandiagonal square, the elements in every row, column, and diagonal, broken and unbroken, have the same sum.) Its magic constant is 870.

8	1	6	121	120	125	29	36	31	132	133	128
3	5	7	126	122	118	34	32	30	127	131	135
4	9	2	119	124	123	33	28	35	134	129	130
69	64	71	98	93	94	40	45	38	83	88	87
79	68	66	91	95	99	39	41	43	90	86	82
65	72	67	96	97	92	44	37	42	85	84	89
116	109	114	13	12	17	137	144	139	24	25	20
111	113	115	18	14	10	142	140	138	19	23	27
112	117	110	11	16	15	141	136	143	26	21	22
105	100	107	62	57	58	76	81	74	47	52	51
106	104	102	55	59	63	75	77	79	54	50	46
101	108	103	60	61	56	80	73	78	49	48	53

FIGURE 2. *A Pandiagonal 12-by-12 Magic Square*

Since each of the 880 fourth-order basic magic squares can be used as a foundation (labelling) square, and each small grid can be filled in eight ways, $8^{16}(880)$ or 247 69797 95053 77280 distinct twelfth-order magic squares (exclusive of rotations and reflections) can be constructed in this manner from the first 144 positive integers.

A larger family of twelfth-order magic squares can be constructed by first taking any nine fourth-order magic squares (repetition permitted) from the 880 squares listed by Benson and Jacoby [1]. In Figure 3, the squares have been ordered by their upper left elements.

The first square in Figure 3 is square (1) in Figure 4. To each element of the second square add 16 to form square (2), to each element of square three add $2 \cdot 16$ to form square (3), and continue the process until the addition of $8 \cdot 16$ to the elements of the ninth square forms square (9).

To construct the twelfth-order magic squares, divide a 12-by-12 grid into nine 4-by-4 grids, thus forming a 3-by-3 grid of grids. Label this 3-by-3 grid with the corresponding elements of the basic third-order magic square (1) in Figure 1. In each 4-by-4 grid place the derived square from Figure 4, in any of its eight orientations, that has the same identification number as the gird. The result is the twelfth-order magic square in Figure 5.

Since any of the 880 fourth-order magic squares can be the basic square for a 4-by-4 grid, and the corresponding derived square can be inserted into the grid in 8 ways, $(880 \cdot 8)^9$ or 4 24770 09370 18688 57788 98944 $\times 10^9$ twelfth-order squares (exclusive of rotations and reflections) can be constructed in this way from the first 144 positive integers.

1	15	14	4	2	5	11	16	3	2	14	15
12	6	7	9	14	9	7	4	13	16	4	1
8	10	11	5	15	8	10	1	12	9	5	8
13	3	2	16	3	12	6	13	6	7	11	10

4	6	9	15	5	2	15	12	6	3	15	10
13	11	8	2	10	16	1	7	4	9	5	16
7	1	14	12	11	13	4	6	13	8	12	1
10	16	3	5	8	3	14	9	11	14	2	7

7	4	14	9	8	5	11	10	9	1	8	16
16	13	3	2	9	12	6	7	14	12	5	3
1	12	6	15	2	3	13	16	4	6	11	13
10	5	11	8	15	14	4	1	7	15	10	2

FIGURE 3. Nine Basic 4-by-4 Magic Squares

(1)				(2)				(3)			
1	15	14	4	18	21	27	32	35	34	46	47
12	6	7	9	30	25	23	20	45	48	36	33
8	10	11	5	31	24	26	17	44	41	37	40
13	3	2	16	19	28	22	29	38	39	43	42

(4)				(5)				(6)			
52	54	57	63	69	66	79	76	86	83	95	90
61	59	56	50	74	80	65	71	84	89	85	96
55	49	62	60	75	77	68	70	93	88	92	81
58	64	51	53	72	67	78	73	91	94	82	87

(7)				(8)				(9)			
103	100	110	105	120	117	123	122	137	129	136	144
112	109	99	98	121	124	118	119	142	140	133	131
97	108	102	111	114	115	125	128	132	134	139	141
106	101	107	104	127	126	116	113	135	143	138	130

FIGURE 4. Nine Derived 4-by-4 Magic Squares

120	117	123	122	13	8	12	1	87	82	94	91
121	124	118	119	3	10	6	15	81	92	88	93
114	115	125	128	2	11	7	14	96	85	89	84
127	126	116	113	16	5	9	4	90	95	83	86
47	46	34	35	69	66	79	76	105	98	111	104
33	36	48	45	74	80	65	71	110	99	102	107
40	37	41	44	75	77	68	70	100	109	108	101
42	43	39	38	72	67	78	73	103	112	97	106
52	61	55	58	135	143	138	130	29	17	20	32
54	59	49	64	132	134	139	141	22	26	23	27
57	56	62	51	142	140	133	131	28	24	25	21
63	50	60	53	137	129	136	144	19	31	30	18

FIGURE 5. A 12-by-12 Magic Square

Together the two families contain $8^{17}(110)(8 \cdot 110^8 + 1)$ distinct twelfth-order magic squares.

This technique can be employed to produce two families of knth order magic squares from magic squares of the kth and nth orders. If $k = n$, there is one family. Such is the family of 134,217,728 ninth-order magic squares [2].

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COIN TOSSING AND THE r-BONACCI NUMBERS

CARL P. McCARTY

La Salle College, Philadelphia, PA 19141

In this paper we find the probability that a fair coin tossed n times will contain a run of at least r consecutive heads.

Let $X_n = \{x_1x_2 \ldots x_n / x_i \in \{h,t\},\ i = 1, 2, \ldots, n\}$ be the set of 2^n equi-probable outcomes and Y_n^r be the subset of X_n each of whose elements contains a run of at least r consecutive heads. Also, let $a(r,n)$ be the cardinality of Y_n^r. We can construct Y_n^r by noting that each of its elements must fall into one of the following two categories:

(1) $$HA_{n-r}$$

(2) $$W_j tHA_{n-j-1-r}$$

where H is the first run of r consecutive heads to appear when reading from left to right,

A_i is an i-string of any combination of heads and tails,

W_j is a j-string of heads and tails not containing H, and

t is a singleton tail.

Since there are $2^j - a(r,j)$ ways in which W_j can occur, the total number of elements of type (2) is

$$\sum_{j=0}^{n-1-r} [2^j - a(r,j)]2^{n-j-1-r}.$$