$$N(z) \leq \sum_{G \leq \eta \log z} \left( \sum_{q_1 \ldots q_G d_G \leq z} (2^{n-1})^G (q_1 \ldots q_G)^{n-1} \left( \frac{z}{q_1 \ldots q_G d_G} \right)^s \right)$$

where $s > n$ will be chosen. This shows

$$N(z) = 0\left( z^s \sum_{G \leq \eta \log z} 2^{(n-1)G} \sum_{q_1 = 1}^{\infty} \cdots \sum_{q_G = 1}^{\infty} \sum_{q_G = 1}^{\infty} (q_1 \ldots q_G d_G)^{n-1-s} \right)$$

$$= 0\left( z^s \sum_{G \leq \eta \log z} (2^{n-1}\zeta(s + 1 - n))^{G+1} \right) = 0\left( z^s (2^{n-1}\zeta(s + 1 - n))^{\eta \log z} \right).$$

We put $s = n + \varepsilon$ and obtain $N(z) = 0(z^\alpha)$ where

$$\alpha = n + \varepsilon + \eta(\log \zeta(1 + \varepsilon) + (n - 1)\log 2).$$

Choosing $\varepsilon > 0$ and $\eta = \eta(\varepsilon)$, we may obtain

$$\varepsilon + \eta[\log \zeta(1 + \varepsilon) + (n - 1)\log 2] \leq \sigma.$$

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## SOLUTION OF THE RECURRENT EQUATION $u_{n+1} = 2u_n - u_{n-1} + u_{n-3}$


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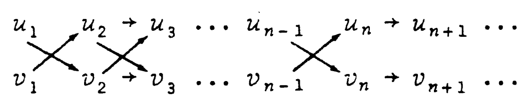
To find the general term of the sequence $\{u_n\}$, we introduce an auxiliary sequence $\{v_n\}$, intertwined with $\{u_n\}$ in the following way:




where

$$(1) \qquad \begin{cases} u_{n+1} = v_{n-1} + u_n, \\ v_{n+1} = u_{n-1} + v_n. \end{cases}$$

It is clear that both sequences are determined as soon as $u_1$, $v_1$ $(= u_3 - u_2)$, and $u_2$, $v_2$ $(= u_4 - u_3)$ are given. $\{u_n\}$ solves our problem since

$$u_{n+1} = v_{n-1} + u_n = u_{n-3} + v_{n-2} + u_n = u_{n-3} + (u_n - u_{n-1}) + u_n.$$

1. Adding the equations in (1) memberwise, we obtain:

$$u_{n+1} + v_{n+1} = (u_{n-1} + v_{n-1}) + (u_n + v_n),$$

which implies that $\{u_n + v_n\}$ *is a Fibonacci sequence* $\{F_n\}$ whose first two terms are

$$u_1 + v_1 \ (= u_1 - u_2 + u_3) \quad \text{and} \quad u_2 + v_2 \ (= u_2 - u_3 + u_4).$$

2. Our problem would be completely solved if we would have an expression for $u_n - v_n = \varepsilon_n$. Subtracting the equations in (1) memberwise, we obtain:

$$\begin{aligned} \varepsilon_{n+1} &= \varepsilon_n - \varepsilon_{n-1}, \\ &= (\varepsilon_{n-1} - \varepsilon_{n-2}) - \varepsilon_{n-1} \quad \text{(replacing } n \text{ by } n - 1 \text{ above)}, \\ &= -\varepsilon_{n-2}, \\ &= -(-\varepsilon_{n-5}) \quad \text{(replacing } n \text{ by } n - 3 \text{ above)}, \\ &= \varepsilon_{n-5}. \end{aligned}$$

Thus, $\{\varepsilon_n\}$ *is a periodic sequence*, with period 6 and

$$\varepsilon_1 = u_1 - v_1 = u_1 + u_2 - u_3,\quad \varepsilon_2 = u_2 - v_2 = u_2 + u_3 - u_4,\quad \varepsilon_3 = \varepsilon_2 - \varepsilon_1,$$
$$\varepsilon_4 = -\varepsilon_1,\qquad \varepsilon_5 = -\varepsilon_2,\qquad \varepsilon_6 = -\varepsilon_3.$$

3. Hence

$$u_n + v_n = F_n$$
$$u_n - v_n = \varepsilon_n = \varepsilon_{[n]} \quad \text{(where } [n] = n \text{ modulo 6)},$$

and

$$\boxed{u_n = \tfrac{1}{2}(F_n + \varepsilon_{[n]})} \qquad (n > 4).$$

Now $F_n$ may be written in the form (using the Binet formula):

$$F_n = (u_1 - u_2 + u_3)N_{n-2} + (u_2 - u_3 + u_4)N_{n-1},$$

where $N_n$ is the integer closest to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

(see, for instance, N. N. Vorob'ev, *Fibonacci Numbers*, Blaisdell Publishing Company, 1961, page 22).

*Remarks*: 1. The method used makes obvious the following relations:

$$u_n + u_{n+3} = \tfrac{1}{2}(F_n + F_{n+3}) = F_{n+2},$$
$$u_{n+6} - u_n = \tfrac{1}{2}(F_{n+6} - F_n) = 2F_{n+3},\ \ldots .$$

2. Any sequence $\{\varepsilon_n\}$ and any Fibonacci sequence are solutions of the given recurrent equation (directly or by our formula).

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## PRIMENESS FOR THE GAUSSIAN INTEGERS

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Complex numbers of the form $a + bi$, where $a$ and $b$ are integers, are commonly called Gaussian Integers. It can be shown that the Gaussian Integers, denoted by $G$, along with addition and multiplication of complex numbers, form an integral domain. One might suspect that many properties about the integers, denoted by $Z$, carry over to $G$. This is indeed the case, and it is the purpose of this paper to examine the property of primeness in the Gaussian domain. The Fundamental Theorem of Arithmetic states that every integer is either a prime or can be uniquely factored into a product of primes, apart from the order in which the factors appear. This theorem also holds for $G$. It is also true that both $G$ and $Z$ are unique factorization domains. For $Z$, the units are 1 and -1, while the units for $G$ are 1, -1, $i$, and $-i$. The job at hand, then, is to determine what elements of $G$ are prime.

For each $\alpha \in G$, $\alpha \cdot \bar{\alpha}$, where $\bar{\alpha}$ is the conjugate of $\alpha$, is called *the norm of* $\alpha$ and is denoted by $N(\alpha)$. Thus for $a,b \in Z$, $N(a + bi) = (a + bi)(a - bi) = a^2 + b^2$. It also follows that for $\alpha,\beta \in G$, $N(\alpha \cdot \beta) = N(\alpha) \cdot N(\beta)$.

Since $G$ is a unique factorization domain, any $\alpha \in G$ can be factored into a product of primes. Therefore, suppose $\alpha = p_1 \cdot p_2 \cdot \ldots \cdot p_n$, where the $p_i$'s ($i = 1, 2, \ldots, n$) are prime in $G$. We thus have $N(\alpha) = N(p_1) \cdot N(p_2) \cdot \ldots \cdot N(p_n)$. Hence, any factorization of $\alpha \in G$ leads to a corresponding factorization of $N(\alpha)$ in $Z$. As a result, $\alpha$ is prime in $G$ if $N(\alpha)$ is prime in $Z$. As an illustration of these results, consider $\alpha = 3 + 7i$. Since $N(\alpha) = 9 + 49 = 58 = 2 \cdot 29$, $3 + 7i$ has at most two prime factors having norms 2 and 29. Those elements of $G$ with norm 2 are $1 \pm i$. Selecting $1 + i$ and solving the equation $(3 + 7i) = (1 + i)(x + iy)$ for $x$ and $y$, one discovers that $(3 + 7i) = (1 + i)(5 + 2i)$. If $1 - i$ were chosen, $3 + 7i = (1 - i)(-2 + 5i)$. This appears at first glance to be a different factorization, but observe that $(3 + 7i) = -i(1 - i)(5 + 2i)$ where $-i$ is a unit. Note also that $N(5 + 2i) = 29$. Hence, $(1 + i)(5 + 2i)$ is a prime factorization of $3 + 7i$.

We now have a procedure for determining whether a Gaussian integer of the form $a + bi$, $a,b \neq 0$, is prime in $G$. What remains is to find a method for determining whether or not