TABLE 2. Values of $B'(n,-1)$

n	$B'(n,-1)$
-1	1
0	0
1	1
2	1
3	4
4	11
5	41
6	162
7	715

Apparently the Bell triangle cannot be extended further because $B(-1,0) = B_{-1}$ which is undefined, by equation (1). Epstein [3] drops the term $0^n/0!$ in equation (1) without explanation and therefore gets $B_0 = 1 - 1/e$, in contradiction with Williams [5], Bell [1], and Rota [4].

The Bell numbers have combinatoric significance in that B_n is the number of ways of factoring a product of n distinct primes. Whether the rest of the numbers in the Bell triangle have any such significance remains to be seen.

REFERENCES

1. E. T. Bell. "Exponential Numbers." *American Math. Monthly* 41 (1934):411-419.
2. Martin Cohn, et al. "On the Number of Partitionings of a Set of n Distinct Objects." *American Math. Monthly* 69 (1962):782-785.
3. Leo Epstein. "A Function Related to the Series e^{e^x}." *J. Math. and Physics* 18 (1939): 153-173.
4. Gian-Carlo Rota. "The Number of Partitions of a Set." *American Math. Monthly* 71 (1964):498-504.
5. G. T. Williams. "Numbers Generated by the Function e^{e^x-1}." *American Math. Monthly* 52 (1945):323-327.

*****

THE EQUATIONS $z^2 - 3y^2 = -2$ AND $z^2 - 6x^2 = -5$

MANORANJITHAM VELUPPILLAI

Royal Holloway College, Egham, Surrey, England

The four numbers 2, 4, 12, 420 have the property that the product of any two increased by 1 is a perfect square. The object of this paper is to prove that no positive integer can replace 420.

Any integer N which can replace 420 while preserving this property must satisfy the equations

$$2N + 1 = x^2, \quad 4N + 1 = y^2, \quad 12N + 1 = z^2.$$

Eliminating N, we have

$$z^2 - 3y^2 = -2 \quad \text{and} \quad z^2 - 6x^2 = -5.$$

Now, the equation $z^2 - 3y^2 = -2$ can be written in the form

$$u^2 - 3v^2 = 1 \tag{1}$$

where $u = z^2 + 1$, $v = zy$.

Substituting for z^2 in $z^2 - 6x^2 = -5$, we have

$$X^2 = 6u + 24 \tag{2}$$

where $X = 6x$.

Hence, to solve the equations of the title, it is sufficient to solve (1) and (2) simultaneously.

Now, all the positive integral solutions of (1) are given by the formula:

(3) $$u_n + \sqrt{3}V_n = (2 + \sqrt{3})^n$$

By (3), we have

$$u_n = \frac{\alpha^n + \beta^n}{2} \quad \text{and} \quad v_n = \frac{\alpha^n - \beta^n}{2\sqrt{3}}$$

where $\alpha = 2 + \sqrt{3}$ and $\beta = 2 - \sqrt{3}$. We have the following equations and congruences:

(4) $u_{-n} = u_n$,

(5) $v_{-n} = v_n$,

(6) $u_{m+n} = u_m u_n + 3v_m v_n$,

(7) $v_{m+n} = u_m v_n + v_m u_n$,

(8) $u_{2n} = 2u_n^2 - 1$,

(9) $v_{2n} = 2u_n v_n$,

(10) $u_{3n} = u_n \cdot f_1(u_n)$,

(11) $v_{3n} = v_n \cdot f_2(u_n)$,

(12) $u_{5n} = u_n \cdot f_3(u_n)$,

(13) $v_{5n} = v_n \cdot f_4(u_n)$,

(14) $u_{7n} = u_n \cdot f_5(u_n)$,

(15) $v_{7n} = v_n \cdot f_6(u_n)$,

(16) $u_{9n} = u_n \cdot f_1(u_n) \cdot f_7(u_n)$,

(17) $v_{9n} = v_n \cdot f_2(u_n) \cdot f_8(u_n)$,

(18) $u_{15n} = u_n \cdot f_1(u_n) \cdot f_3(u_n) \cdot f_9(u_n)$

(19) $v_{15n} = v_n \cdot f_2(u_n) \cdot f_4(u_n) \cdot f_{10}(u_n)$,

(20) $u_{n+2r} \equiv u_n \pmod{v_r}$,

(21) $u_{n+2r} \equiv -u_n \pmod{u_r}$,

where

$f_1(u_n) = 4u_n^2 - 3$,

$f_2(u_n) = 4u_n^2 - 1$,

$f_3(u_n) = 16u_n^4 - 20u_n^2 + 5$,

$f_4(u_n) = 16u_n^4 - 12u_n^2 + 1$

$f_5(u_n) = 64u_n^6 - 112u_n^4 + 56u_n^2 - 7$,

$f_6(u_n) = 64u_n^6 - 80u_n^4 + 24u_n^2 - 1$,

$f_7(u_n) = 64u_n^6 - 96u_n^4 + 36u_n^2 - 3$,

$f_8(u_n) = 64u_n^6 - 96u_n^4 + 36u_n^2 - 1$,

$f_9(u_n) = 256u_n^8 - 448u_n^6 + 224u_n^4 - 32u_n^2 + 1$,

$f_{10}(u_n) = 256u_n^8 - 576u_n^6 + 416u_n^4 - 96u_n^2 + 1$.

We now have the following table of values:

n	u_n	v_n
0	1	0
1	2	1
2	7	4
3	26	15
4	97	56
5	362	209
6	1351	780
7	5042	2911
8	18817	10864
9	70226	40545
10	262087	151316
11	978122	564719
12	3650401	2107560
13	13623482	7865521

We note that both z and y are odd and hence u is even and v is odd. Hence, we have to consider only the odd values of n.

The proof is now accomplished in eleven stages:

(i) (2) is impossible if $n \equiv 3 \pmod 6$.
For, $u_n \equiv 0 \pmod{13}$ and then $X^2 \equiv -2 \pmod{13}$ and since $(-2|13) = -1$, (2) is impossible.

(ii) (2) is impossible if $n \equiv 5 \pmod{10}$.
For, using (20), $u_n \equiv u_5 \pmod{v_5} \equiv 362 \pmod{209} \equiv -1 \pmod{11}$. But then $X^2 \equiv 7 \pmod{11}$ and $(7|11) = -1$ and hence (2) is impossible.

(iii) (2) is impossible if $n \equiv \pm 5 \pmod{14}$.
For, $u_n \equiv u_{\pm 5} \pmod{v_7} \equiv u_5 \pmod{v_7}$, using (4). Now, $71|v_7$, $u_5 \equiv 7 \pmod{71}$ and then $X^2 \equiv -5 \pmod{71}$. Since $(-5|71) = -1$, (2) is impossible.

(iv) (2) is impossible if $n \equiv \pm 3 \pmod{20}$.
For, using (21), $u_n \equiv u_{\pm 3} \equiv \pm u_3 \pmod{u_{10}}$ and then $X^2 \equiv 180$ or $-132 \pmod{7 \cdot 37441}$. Now, since $(180|7) = -1$ and $(-132|37441) = -1$, (2) is impossible.

(v) (2) is impossible if $n \equiv \pm 3, \pm 11, \pm 13 \pmod{28}$.
For, when $n \equiv \pm 11 \pmod{28}$, using (4) and (20) we have $u_n \equiv u_{11} \pmod{v_{14}}$. Now, $2521 | v_{14}$ and $u_{11} \equiv -26 \pmod{2521}$. But then $X^2 \equiv -132 \pmod{2521}$ and since $(-132|2521) = -1$, this is impossible.
When $n \equiv \pm 3, \pm 13 \pmod{28}$, using (4) and (21) we have, $u \equiv \pm u$, $\pm u_{13} \pmod{u_{14}}$. Now, 7, 337, $3079 | u_{14}$ and $u_3, u_{13} \equiv 5 \pmod 7$, $u_3 \equiv 26 \pmod{337}$ and $u_{13} \equiv 1986 \pmod{3079}$.
Hence, $X^2 \equiv 24 + 6u_3$, $X^2 \equiv 24 + 6u_{13}$ are impossible modulo 7, $X^2 \equiv 24 - 6u_3$ is impossible modulo 337, and $X^2 \equiv 24 - 6u_{13}$ is impossible modulo 3079.

(vi) (2) is impossible if $n \equiv \pm 11, \pm 13 \pmod{30}$.
For, $u_n \equiv u_{11}, u_{13} \pmod{v_{15}}$. Now $29 | v_{15}$ and $u_{11} \equiv 10 \pmod{29}$ and $u_{13} \equiv 7 \pmod{29}$. Hence, $X^2 \equiv -3 \pmod{29}$ and $X^2 \equiv 8 \pmod{29}$ and since $(-3|29) = -1$, $(8|29) = -1$, both are impossible.

(vii) (2) is impossible if $n \equiv \pm 13 \pmod{42}$.
For, $u_n \equiv u_{13} \pmod{v_{21}}$ and then $X^2 \equiv 24 + 6u_{13} \pmod{v_{21}}$. Now $2017 | v_{21}$ and $X^2 = 1991 \pmod{2017}$, and since $(1991|2017) = -1$, (2) is impossible.

(viii) (2) is impossible if $n \equiv \pm 21 \pmod{70}$. For, $u_n \equiv u_{21} \pmod{v_{35}}$.

$$v_{35} = v_{7\cdot 5} = v_5(8u_5 - 4u_5 - 4u_5 + 1)(8u_5 + 4u_5 - 4u_5 - 1)$$
$$= v_5 \cdot v_7 \cdot 9243361 \cdot 5352481.$$

Also, $u_{21} = u_7(4u_7^2 - 3)$
$$x^2 = 24 + 6u_7(4u_7^2 - 3) \pmod{5352481}$$
$$\equiv -305121648 \qquad \pmod{5352481}.$$

Now, $$\left(\frac{-305121648}{5352481}\right) = \left(\frac{2}{5352481}\right)^4\left(\frac{3}{5352481}\right)\left(\frac{6356701}{5352481}\right) = \left(\frac{1004220}{5352481}\right) = \left(\frac{797}{5352481}\right)$$
$$= \left(\frac{-171}{797}\right) = \left(\frac{113}{171}\right) = \left(\frac{29}{113}\right) = \left(\frac{-3}{29}\right) = -1.$$

Hence, (viii) is impossible.

(ix) (2) is impossible if $n \equiv \pm 29, \pm 31 \pmod{90}$.
For, $u_n \equiv u_{29}, u_{31} \pmod{v_{45}}$. Now $83609 | v_{45}$ and $u_{29} = 2u_{30} - 3v_{30} = 2u_{10}(4u_{10}^2 - 3) - 3v_{10}(4u_{10}^2 - 1) \equiv 9253 \pmod{83609}$. Hence, $X^2 \equiv 55542 \pmod{83609}$ and since $(55542|83609) = -1$, (2) is impossible.
Also, $17 | v_{45}$ and $u_{31} = 2u_{30} + 3v_{30} \equiv 5 \pmod{17}$ and hence $X^2 \equiv 3 \pmod{17}$. Since $(3|17) = -1$, (2) is impossible.

(x) (2) is impossible if $n \equiv \pm 1 \pmod{252}$, $n \neq \pm 1$.
For, we can write $n = \pm 1 + 63k(2\ell + 1)$, where ℓ is an integer and $k = 2^t$, $t \geq 2$.
Then, $u_n \equiv \pm u_{\pm 1 + 63k} \equiv \pm 3v_{63k} \pmod{u_{63k}}$.
Now, $v_{63k} = v_{9\cdot 7k} \equiv v_{7k} \pmod{u_{7k}} \equiv v_k(32u_k^4 - 32u_k^2 + 6)\ [\text{mod } f_5(u_k)]$

And, $v_{63k} = v_{7\cdot 9k} \equiv -v_{9k} \pmod{u_{9k}} \equiv -2v_k(4u_k^2 - 1)\ [\text{mod } f_7(u_k)]$

Hence, $X^2 \equiv 24 \pm 18v_k(32u_k^4 - 32u_k^2 + 6)\ [\text{mod } f_5(u_k)]$
$$\equiv 24 \mp 36v_k(4u_k^2 - 1)\ [\text{mod } f_7(u_k)].$$

First, consider $X^2 \equiv 24 + 18v_k(32u_k^4 - 32u_k^2 + 6)\ [\text{mod } f_5(u_k)]$.

Now, $$\left(\frac{24 + 18v_k(32u_k^4 - 32u_k^2 + 6)}{f_5(u_k)}\right) = \left(\frac{24 + 18v_k(288v_k^4 + 96v_k^2 + 6)}{1728v_k^6 + 720v_k^4 + 72v_k^2 + 1}\right)$$
$$= \left(\frac{144v_k^4 + 36v_k^2 - 8v_k^2 + 1}{\frac{1}{2}(432v_k^5 + 144v_k^3 + 9v_k + 2)}\right)$$
$$= \left(\frac{36v_k^3 + 24v_k^2 + 6v_k + 2}{144v_k^4 + 36v_k^2 - 8v_k + 1}\right)$$

(continued)

$$= \left(\frac{3}{\frac{1}{2}(36v_k^3 + 24v_k^2 + 6v_k + 2)}\right)\left(\frac{228v_k^2 + 19}{\frac{1}{2}(36v_k^3 + 24v_k^2 + 6v_k + 2)}\right)$$

$$= (-)\left(\frac{36v_k^3 + 24v_k^2 + 6v_k + 2}{19}\right)$$

Similarly, $\left(\frac{24 - 18v_k(32u_k^4 - 32u_k^2 + 6)}{f_5(u_k)}\right) = \left(\frac{36v_k^3 - 24v_k^2 + 6v_k - 2}{19}\right)$

Next, consider $X^2 \equiv 24 \mp 36v_k(4u_k^2 - 1)$ [mod $f_7(u_k)$].

Now, $\left(\frac{24 - 36v_k(4u_k^2 - 1)}{f_7(u_k)}\right) = \left(\frac{24 - 36v_k(12v_k^2 + 3)}{1728v_k^6 + 864v_k^4 + 108v_k^2 + 1}\right) = \left(\frac{1728v_k^6 + 864v_k^4 + 108v_k^2 + 1}{\frac{1}{2}(36v_k^3 + 9v_k - 2)}\right)$

$$= \left(\frac{96v_k^3 + 24v_k + 1}{\frac{1}{2}(36v_k^3 + 9v_k - 2)}\right) = \left(\frac{36v_k^3 + 9v_k - 2}{19}\right)$$

Similarly, $\left(\frac{24 + 36v_k(4u_k^2 - 1)}{f_7(u_k)}\right) = (-)\left(\frac{36v_k^3 + 9v_k + 2}{19}\right)$

The residues of v_k, $36v_k^3 \pm 24v_k^2 \pm 6v_k + 2$ and $36v_k^3 + 9v_k \pm 2$ modulo 19 are periodic and the length of the period is 4. The following table gives these residues and the signs of $(24 \pm 18v_k(32u_k^4 - 32u_k^2 + 6)|f_5(u_k))$ and $(24 \mp 36v_k(4u_k^2 - 1)|f_7(u_k))$.

$k = 2^t$	$t = 2$	3	4	5	6
v_k (mod 19)	-1	-4	1	4	-1
$36v_k^3 + 24v_k^2 + 6v_k + 2$ (mod 19)	3	-4	-8	-3	
$36v_k^3 - 24v_k^2 + 6v_k - 2$ (mod 19)	8	3	-3	4	
$36v_k^3 + 9v_k + 2$ (mod 19)	-5	-1	9	5	
$36v_k^3 + 9v_k - 2$ (mod 19)	-9	-5	5	1	
$(24 + 18v_k(32u_k^4 - 32u_k^2 + 6)\|f_5(u_k))$	+1	+1	-1	-1	
$(24 - 36v_k(4u_k^2 - 1)\|f_7(u_k))$	-1	-1	+1	+1	
$(24 - 18v_k(32u_k^4 - 32u_k^2 + 6)\|f_5(u_k))$	-1	-1	+1	+1	
$(24 + 36v_k(4u_k^2 - 1)\|f_7(u_k))$	+1	+1	=1	-1	

From the above table, we see that the congruences $X^2 \equiv 24 + 18v_k(32u_k^4 - 32u_k^2 + 6)$ [mod $f_5(u_k)$] and $X^2 \equiv 24 - 36v_k(4u_k^2 - 1)$ [mod $f_7(u_k)$] cannot hold simultaneously, and the congruences $X^2 \equiv 24 - 18v_k(32u_k^4 - 32u_k^2 + 6)$ [mod $f_5(u_k)$] and $X^2 \equiv 24 + 36v_k(4u_k^2 - 1)$ [mod $f_7(u_k)$] cannot hold simultaneously.

Hence, (2) is impossible.

(xi) $n \equiv \pm 7 \pmod{60}$; $n \neq \pm 7$ is impossible.

For, we can write $n = \pm 7 + 2.15k\ell$, where $k = 2^t$, $t \geq 1$ and ℓ is an odd integer. Then, by applying (21) ℓ times, we have

$$u_n \equiv -u_7 \pmod{u_{15k}} \equiv -5042 \pmod{u_k \cdot f_1(u_k) \cdot f_3(u_k) \cdot f_9(u_k)}$$

Hence, $X^2 \equiv 24 - 6.5042 \equiv -30228 \pmod{u_k \cdot f_1(u_k) \cdot f_3(u_k) \cdot f_9(u_k)}$. Note that when $t = 1$, $u_n \equiv -2 \pmod 7$ and then $X^2 \equiv 5 \pmod 7$ and $(5|7) = -1$.

When $t \geq 2$, we have

$$(-30228|u_k) = (u_k|11)(u_k|229) = (-)(u_k|229) \text{ when } u_k \equiv -4 \pmod{11}$$
$$= (u_k|229) \quad \text{when } u_k \equiv -2 \pmod{11};$$

$$(-30228|f_1(u_k)) = (-)(f_1(u_k)|229);$$

$$(-30228|f_3(u_k)) = (-)(f_3(u_k)|229) \text{ when } u_k \equiv -4 \pmod{11}$$
$$= (f_3(u_k)\ 229) \quad \text{when } u_k \equiv -2 \pmod{11};$$
$$(-30228|f_9(u_k)) = (-)(f_9(u_k)|229).$$

The residues of u_k, $f_1(u_k)$, $f_3(u_k)$, and $f_9(u_k)$ modulo 229 are periodic and the length of the period is 9. The following table gives the values of these residues and the signs of $(-30228|u_k)$, $(-30228|f_1(u_k))$, $(-30228|f_3(u_k))$ and $(-30228|f_5(u_k))$.

$k = 2^t$	$t = 2$	3	4	5	6	7	8	9	10	11
u_k (mod 229)	97	39	64	-53	121	-31	89	40	-7	97
$f_1(u_k)$ (mod 229)	77	127	122	12	-63	177	79	-15	193	
$f_3(u_k)$ (mod 229)	51	-4	-109		12	132	-93			
$f_9(u_k)$ (mod 229)	103				159		58			
	when $u_k \equiv -4 \pmod{11}$									
$(-30228\ u_k)$	-1	+1	-1	-1	-1	+1	+1	+1	+1	
$(-30228\ f_1(u_k))$		+1				+1	+1	-1	-1	
$(-30228\ f_3(u_k))$		-1				-1	+1			
$(-30228\ f_9(u_k))$							-1			
	when $u_k \equiv -2 \pmod{11}$									
$(-30228\ u_k)$	+1	-1	+1	+1	+1	-1	-1	-1	-1	
$(-30228\ f_1(u_k))$	+1		+1	-1	+1					
$(-30228\ f_3(u_k))$	+1		-1		+1					
$(-30228\ f_9(u_k))$	-1				-1					

Hence, (2) is impossible.

Summarizing the results, we see that (1) and (2) can hold for n odd, only for $n = 1$ and $n = 7$, and these values do indeed satisfy with $u = 2$, $v = 1$, $x = 1$, and $u = 5042$, $v = 2911$, $x = 29$. $x = 1$ gives the trivial solution $N = 0$ and $x = 29$ gives the solution $N = 420$.

*****

GENERATION OF FIBONACCI NUMBERS BY DIGITAL FILTERS

SALAH M. YOUSIF
California State University, Sacramento, California

ABSTRACT

This paper presents some applications of Fibonacci numbers in system and communication theory. Methods of generating Fibonacci sequences and codes by sequential binary filters are given.

INTRODUCTION

The role that Fibonacci numbers play in system theory is worthy of engineering investigations. Fibonacci numbers find their way in algebraic coding theory in communications, linear sequential circuits, and linear digital filters. Although some of these applications are not direct realizations of Fibonacci numbers, they provide the conceptual framework for the related model. For example, the concept of recurrence equation that generates the numbers is utilized to generate difference codes which are used in radar ranging by long-range radars, such as satellite tracking radars and radars that are used for planet's ranging [2]. Another example of Fibonacci numbers is one used to generate a model for population growth in