Proof of Theorem 2: Initially, consider the case where N is odd and $x_1 < y_1$. The remaining cases are proved in a similar manner. Using the addition formulas (4) for sin $N\theta$ and cos $N\theta$ and Lemma 1, the following values are obtained for the sides of T_N in terms of the generators of T_1 and T_{N-1}:

$$x_N = 4m_{N-1}n_{N-1}m_1n_1 + m_{N-1}^2m_1^2 - m_{N-1}^2n_1^2 - n_{N-1}^2m_1^2 + n_{N-1}^2n_1^2$$

$$y_N = 2[m_1n_1(m_{N-1}^2 - n_{N-1}^2) - m_{N-1}n_{N-1}(m_1^2 - n_1^2)]$$

$$z_N = m_{N-1}^2m_1^2 + m_{N-1}^2n_1^2 + n_{N-1}^2m_1^2 + n_{N-1}^2n_1^2$$

Consequently:

$$m_N = \sqrt{(z_N + x_N)/2} = m_1m_{N-1} + n_1n_{N-1}$$

$$n_N = \sqrt{(z_N - x_N)/2} = m_1n_{N-1} - n_1m_{N-1}$$

It is also to be noted that the sides of T_N serve as generators for T_{2N} where these exist. Thus, for instance, for $T_1 = (5,12,13)$, the sides 5 and 12 serve as generators for $T_2 = (119,120,169)$. Similarly, for $T_2 = (1081,840,1369)$, the sides serve as generators for $T_4 = (462961,1816080,1874161)$.

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PROOF THAT THE AREA OF A PYTHAGOREAN TRIANGLE IS NEVER A SQUARE

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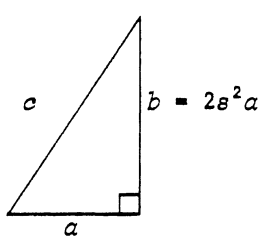
Prove that the area of an integral-sided (Pythagorean) triangle is never a square integer. In the diagrams provided below, the two triangles are equivalent. Thus, $a = a$, $b = n$, and $c = (n + k)$, where a, b, n, and k as well as s are integers. A = the area of the triangles.

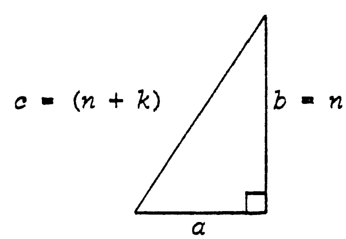


$A = \frac{1}{2}(2s^2a)a = s^2a^2$, which is a square

$a^2 + b^2 = c^2$

$a^2 + (2s^2a)^2 = c^2$

$a^2 + 4s^4a^2 = c^2$



$a^2 + b^2 = c^2$

$a^2 + n^2 = (n + k)^2;\quad a^2 = 2kn + k^2$

$(2kn + k^2) + n^2 = (n + k)^2$

$$a^2 + b^2 = c^2 \quad \text{(Pythagorean Theorem)}$$

$$a^2 + n^2 = (n + k)^2 \quad \text{(Pythagorean Theorem and equivalence of above diagrams)}$$

$$a^2 = 2kn + k^2$$

$$b = 2s^2a \quad \text{(from above diagrams)}$$

$$b^2 = n^2 = 4s^4(a^2) \quad (\text{since } b = n \text{ and } b = 2s^2a)$$

$$n^2 = 4s^4(2kn + k^2) \quad (\text{since } a^2 = 2kn + k^2)$$

$$n^2 = 8ks^4n + 4k^2s^4$$

$$n^2 - 8ks^4n - 4k^2s^4 = 0$$

$$n = \frac{8ks^4 \pm \sqrt{64k^2s^8 - 4(-4k^2s^4)}}{2}$$

$$n = \frac{8ks^4 \pm \sqrt{64k^2s^8 + 16k^2s^4}}{2}$$

$$n = \frac{8ks^4 \pm \sqrt{16k^2s^4(4s^4 + 1)}}{2}$$

$$n = \frac{8ks^4 \pm 4ks^2\sqrt{4s^4 + 1}}{2}$$

$$n = 4ks^4 + 2ks^2\sqrt{4s^4 + 1}$$

From the above, we obtain $a^2 = 2kn + k^2$, $b^2 = n^2$, $c^2 = (n + k)^2$.

If n is irrational for all integral values of a, b, c, n, and k, then a^2, b^2, and c^2 cannot all be squares. If a^2, b^2, and c^2 are not squares, then a, b, and c are not integers, and the triangle is not an integral-sided, or Pythagorean, triangle. n can be an integer only if $\sqrt{4s^4 + 1}$ is an integer, and $\sqrt{4s^4 + 1}$ is an integer only if $s^4 = 0$—that is to say, if $s = 0$. From the diagrams, you can see that when $s = 0$, $b = 0$, and since the area of a triangle = $\frac{1}{2}ab$, this triangle has an area of 0.

Thus, dismissing the case when the area of the triangle is 0, the area of an integral-sided right triangle is never a square number.

This proof centers around the assumption that for integers a, n, and k, $a^2 + n^2 = (n + k)^2$. For example, when $a = 3$, $n = 4$, and $k = 1$, $3^2 + 4^2 = (4 + 1)^2$.

The following result—obtained by using a similar approach against Fermat's Last Theorem, where $x^n + y^n \neq z^n$ for integers when $n > 2$—is presented for the interest of the reader. For $n = 3$, $a^3 + n^3 = (n + k)^3$. Thus,

$$a^3 = 3kn^2 + 3k^2n + k^3$$

$$3kn^2 + 3k^2n + k^3 - a^3 = 0$$

$$n = \frac{-3k^2 \pm \sqrt{9k^4 - 4(3k)(k^3 - a^3)}}{6}$$

$$n = \frac{-3k^2 \pm \sqrt{12a^3k - 3k^4}}{6k}$$

I am not sure whether or not this result is of any use, or if it can be generalized for powers greater than the third power, but I intend to pursue this line of reasoning.

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