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A DIVISIBILITY PROPERTY OF BINOMIAL COEFFICIENTS

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Let p be a prime number. Let the integers $a_{n\ell}$ be defined by the identity

$$\binom{py}{n} = \sum_{\ell} a_{n\ell}\binom{y}{\ell}.$$

The purpose of this note is to prove that the exponent to which p divides $a_{n\ell}$ is at least $\ell - (n - \ell)/(p - 1)$.

Let Y be a set with y elements. Let $Y_1, \ldots, Y_p$ be disjoint sets, each equipped with a fixed bijection to Y. We wish to count the subsets N of $Y_1 \cup \cdots \cup Y_p$ having exactly n elements. For such a setset N, denote by N_i the image of $N \cap Y_i$ in Y.

If j is an m-tuple $(i_1, \ldots, i_m)$ with $1 \le i_1 < i_2 < \cdots < i_m \le p$, write $i \in \operatorname{supp} j$ if $i = i_k$ for some k.

Let $S_j^m = \{x \in \cup N_i \mid x \in N_i \text{ if and only if } i \in \operatorname{supp} j\}$. The sets S_j^m are pair-wise disjoint, and $N_i = \cup\{S_j^m \mid i \in \operatorname{supp} j\}$. Moreover, it is easily seen that any change in the ordered p-tuple $(N_1, \ldots, N_p)$ of subsets must change some S_j^m. So producing the sets $N_1, \ldots, N_p$ is the same as producing the sets S_j^m.

Let $L = \cup N_i$, and let ℓ be its cardinality. Let $S^m = \bigcup_j S_j^m$; then S^m consists of the points of L that correspond to exactly m points of N. If t_m is the cardinality of S^m, therefore, one has $n = \ell + \sum_{m=2}^{p} (m - 1)t_m$, and $n/p \le \ell \le n$.

We construct as follows. First select a subset L of Y with cardinality ℓ between n/p and n. Then select a subset S^p of L with cardinality t_p at most $(p - 1)^{-1}(n - \ell)$. Then select a subset S^{p-1} of $L - S^p$ with cardinality t_{p-1} at most $(p - 2)^{-1}(n - \ell - (p - 1)t_p)$. Continue in this way until S^3 has been selected as a subset of $L - S^p - \cdots - S^4$ with cardinality t_3 at most $2^{-1}(n - \ell - (p - 1)t_p - \cdots - 3t_4)$. Now select a subset S^2 of $L - S^p - \cdots - S^3$ with cardinality t_2 equal to

$$n - \ell - \sum_{m=3}^{p} (m - 1)t_m.$$

Define $S^1 = L - S^p - \cdots - S^2$ with cardinality t_1. Finally, select a partition of each S^m into $\binom{p}{m}$ subsets S_j^m.

The above procedure yields the following expression for $\binom{py}{n}$:

$$\sum_{\ell}\binom{y}{\ell}\sum_{t_p}\binom{\ell}{t_p}\sum_{t_{p-1}}\binom{\ell - t}{t_{p-1}} \cdots \binom{\ell - t_p - \cdots - t_3}{t_2}\binom{p}{1}^{t_1} \cdots \binom{p}{p-1}^{t_{p-1}},$$

in which the numbers ℓ and t_m are constrained by the equalities and inequalities of the preceding paragraph. In this expression, each term in the coefficient of $\binom{y}{\ell}$ includes a power of p at least $t_1 + \cdots + t_{p-1} = \ell - t_p \ge \ell - (p - 1)^{-1}(n - \ell)$.

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