(17) $$C(Z) = 1 + Z + Z^2.$$

$C(Z)$ is primitive of degree 2. Therefore, the code generated by $C(Z)$ is periodic with maximal period 3. Long period difference codes of this type are usually used in satellite communications. As an example, the primitive polynomial $1 + x + x^{22}$ generates a code sequence of a period $2^{22} - 1 \simeq 4{,}194{,}393$.

The Fibonacci code sequence over $GF(2)$ has correlation function $R(\ell) = -1$ for all shifts ℓ except for $\ell = 0$ and multiples of $2^2 - 1$, at which the value of $R(\ell)$ is $2^2 - 1 = 3$. The correlation property is of great importance in the ranging operation of satellite radars.

It has been shown that Fibonacci sequences can be used in coding and communication theory, and can be implemented by binary digital filters. Similar applications can utilize this approach to generate Fibonacci numbers.

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THE FIBONACCI SERIES IN THE DECIMAL EQUIVALENTS OF FRACTIONS

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SUMMARY

Four numbers below 100, as denominators of fractions, yield decimal equivalents in which the sequence of digits can also be produced by summations of the terms of the Fibonacci series.

Where every Fibonacci term is used, and moving each term one place to the right, the sequence is that for 1/89; using every second term, the sequence is that for 1/71; with every third term, 2/59; and with every fourth term, 3/31.

The larger denominators: 9899, 9701, 9599, 9301, 8899, 8201, 7099, 6301, and 2399, give repeating decimal equivalents which can be obtained by the summations of every Fibonacci term, every second, third, ..., up to every ninth term, in this case moving each successive term two places to the right. Moreover, the numerators associated with these denominators are: 1, 1, 2, 3, 5, 8, 13, 21, and 34, the first nine terms in the Fibonacci series.

Still larger denominators yield Fibonacci decimal equivalents. Using every fourteenth term, and moving each term three places to the right, the sequence for 377/15701 is obtained.

The decimal equivalents for 9/71, 1/109, 1/10099, and others, can be generated from *right* to *left* by a reverse summation of Fibonacci terms.

The Lucas-, Negative Fibonacci-, Tribonacci-, and other series produce sequences of digits in repeating decimals.

INTRODUCTION

The Fibonacci series is thus defined: $F_1 = 1$; $F_2 = 1$; $F_{(n-1)} + F_n = F_{(n+1)}$; and the first several terms are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... Recently, Brousseau [1] called attention to the fact that the sequence of digits in the decimal equivalent of 1/89 is developed by a summation of the Fibonacci series where each successive term is moved one place to the right; thus,

```
112358
     13
      21
       34
        55
         89
         144
          233
           377
             ...
11235955056 ...
```

This method of summation has been called "diagonalization" by Kaprekar [2]. A better expression for the summation leading to 1/89 is as follows:

$$1/89 = F_1 \cdot 10^{-2} + F_2 \cdot 10^{-3} + F_3 \cdot 10^{-4} + \cdots + F_n \cdot 10^{-(n+1)}$$

or

$$1/89 = \sum_{n=0}^{\infty} F_n \cdot 10^{-(n+1)} = 0.011235955056\ldots .$$

In 1971, Wlodarski [3] showed that the digital sequence for 2/59 was produced by the diagonalization of every third Fibonacci term, starting with the third term.

This paper shows how the Fibonacci series can generate the repeating decimal equivalents of an infinity of fractions.

DISCUSSION

A general expression for the decimal equivalents of certain fractions derivable from the Fibonacci series is

$$1/N = \sum_{n=0}^{\infty} F_{(an+b)} \cdot 10^{-k(n+1)},$$

where $a = 1, 2, 3, \ldots$, and indicates whether every term is used ($a = 1$), or every second term ($a = 2$), every third term ($a = 3$), ...; where $b = 0, 1, 2, 3, \ldots$, and defines further which term is used to start the diagonalization, and where $k = 1, 2, 3, \ldots$, which controls the number of places that each successive term is moved to the right.

In the application of this expression where $a = 1$, $b = 0$, and $k = 1$, the value for 1/89 is obtained. As b takes other values, 1, 2, 3, 4, ..., the denominator remains 89, but the numerators are 1, 10, 11, 21, 32, 53, ..., which appear in a Fibonacci sequence. The reader may wish to check some of these numerators himself.

Where $a = 2$, $b = 0$, and $k = 1$, the sequence is:

```
0.0138
     21
      55
      144
       377
        987
         ...
_____________________
0.0140845... = 1/71
```

Where $a = 3$, $b = 0$, and $k = 1$, the sequence is:

```
0.028
    34
    144
     610
     2554
     10946
         ...
_____________________
0.0338983... = 2/59
```

Where $a = 4$, which means starting with the fourth term and using every fourth term thereafter, a decimal equivalent is not easily obtained, because the diagonalization does not rapidly converge, although it does definitely not diverge. This fraction is inferred to be 3/31 = 0.09774193... from subsequent considerations.

In the diagonalization where $k = 2$, $a = 1, 2, 3, 4, \ldots$, and $b = 0$, the tediously developed decimal equivalents were for the fractions: 1/9899, 1/9701, 2/9599, 3/9301, 5/8899, and 8/8201. For example, 1/9899 = 0.000101020305081321345... . The ..99 and ..01 terminations of these denominators suggested a classification into two groups. They are so arranged in Table 1, bounded by the dashed lines, and their differences were noted also. These differences appeared to be in a Lucas sequence, and when extended outside the dashed lines, they gave rise to additional denominators. The numerators associated with each of them are also tabulated.

The new numbers thus found, 7099, 6301, and 2399, are the denominators of fractions, the decimal equivalents of which can be found by the diagonalization of every seventh, eighth, and ninth Fibonacci term, each term being moved two places to the right. The respective numerators for these three fractions are 13, 21, and 34.

TABLE 1. Differences between Denominators (k = 2)

GROUP I			GROUP II		
Num.	Denom.	Diff.	Num.	Denom.	Diff.
-	10099		-	9801	
		200			100
1	9899		1	9701	
		300			400
2	9599		3	9301	
		700			1100
5	8899		8	8201	
		1800			2900
13	7099		21	5301	
		4700			7600
34	2399			-2299	
		12300			
	-9901				

It was easy to extend this line of reasoning to the case where $a = 3$, meaning that each term is displaced three places to the right. For example,

$$1/998999 = 0.000001001002003005008013021\ldots\ .$$

In Table 2, the denominators within the dashed lines were found by calculation, and by means of their differences all of the other numbers were determined. The numerators associated with them are also included.

TABLE 2. Differences between Denominators (k = 3)

GROUP I			GROUP II		
Num.	Denom.	Diff.	Num.	Denom.	Diff.
-	1000999		-	998001	
		2000			1000
1	998999		1	997001	
		3000			4000
2	995999		3	993001	
		7000			11000
5	988999		8	982001	
		18000			29000
13	970999		21	953001	
		47000			76000
34	923999		55	877001	
		123000			199000
89	800999		144	678001	
		322000			521000
233	478999		377	157001	
		843000			1364000
-	-364001		-	-1206999	

These tabulations suggested a return to the numbers below 100, arranging them in the same way and noting the differences between them, as in Table 3. By analogy with Tables 1 and 2, the number 31 should be reasonably included as a denominator, with 3 as its numerator. The statement in paragraph 2 of the Summary is thus justified, if not rigorously proved.

At the bottoms of Tables 1, 2, and 3, there are some negative numbers that are developed by the application of successive Lucas differences, but these have not been investigated to see how they might relate to the Fibonacci series. At the top right of Tables 1, 2, and 3 are found the numbers 81, 9801, and 998001, which are 9^2, 99^2, and 999^2, respectively. Their relation to the Fibonacci series is not clear.

TABLE 3. Differences between Denominators (k = 1)

GROUP I			GROUP II		
Num.	Denom.	Diff.	Num.	Denom.	Diff.
-	109		-	81	
		20			10
1	89		1	71	
		30			40
2	59		3	31	
		70			110
-	-11		-	-79	

REVERSE FIBONACCI SERIES

Using the Fibonacci series and starting at the right, moving each successive term one place to the left, a reverse Fibonacci diagonalization is obtained:

```
                         13853211
                        21
                       34
                      55
                     89
                   144
                  233
                 377
               ......
1/109 = 0.0091473......8348623853211
```

where 88 intermediate digits have been omitted.

In the same manner, but moving each successive term two places to the left, the decimal equivalent of 1/10099 would ultimately be obtained, where the terminal digits again form a reverse Fibonacci series, and the repeating decimal portion is of undetermined length:

1/10099 = 0.000091473......2113080503020101

Taking every second Fibonacci term and diagonalizing one place to the left, the result is the decimal equivalent of 9/71:

9/71 = 0.12676053......4507042253521

CONCLUSION

There are infinite families of denominators which have repeating decimal equivalents with digital sequences derivable from the Fibonacci series. The numerators of these denominators, as well as the differences between them, also form Fibonacci sequences.

The interested reader might wish to extend the above abbreviated presentation.

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