$$(-30228|f_3(u_k)) = (-)(f_3(u_k)|229) \text{ when } u_k \equiv -4 \pmod{11}$$
$$= (f_3(u_k)\ 229) \quad \text{ when } u_k \equiv -2 \pmod{11};$$
$$(-30228|f_9(u_k)) = (-)(f_9(u_k)|229).$$

The residues of u_k, $f_1(u_k)$, $f_3(u_k)$, and $f_9(u_k)$ modulo 229 are periodic and the length of the period is 9. The following table gives the values of these residues and the signs of $(-30228|u_k)$, $(-30228|f_1(u_k))$, $(-30228|f_3(u_k))$ and $(-30228|f_5(u_k))$.

$k = 2^t$	$t = 2$	3	4	5	6	7	8	9	10	11
u_k (mod 229)	97	39	64	-53	121	-31	89	40	-7	97
$f_1(u_k)$ (mod 229)	77	127	122	12	-63	177	79	-15	193	
$f_3(u_k)$ (mod 229)	51	-4	-109		12	132	-93			
$f_9(u_k)$ (mod 229)	103				159		58			
	when $u_k \equiv -4 \pmod{11}$									
$(-30228\ u_k)$	-1	+1	-1	-1	-1	+1	+1	+1	+1	
$(-30228\ f_1(u_k))$		+1				+1	+1	-1	-1	
$(-30228\ f_3(u_k))$		-1				-1	+1			
$(-30228\ f_9(u_k))$							-1			
	when $u_k \equiv -2 \pmod{11}$									
$(-30228\ u_k)$	+1	-1	+1	+1	+1	-1	-1	-1	-1	
$(-30228\ f_1(u_k))$	+1		+1	-1	+1					
$(-30228\ f_3(u_k))$	+1		-1		+1					
$(-30228\ f_9(u_k))$	-1				-1					

Hence, (2) is impossible.

Summarizing the results, we see that (1) and (2) can hold for n odd, only for $n = 1$ and $n = 7$, and these values do indeed satisfy with $u = 2$, $v = 1$, $x = 1$, and $u = 5042$, $v = 2911$, $x = 29$. $x = 1$ gives the trivial solution $N = 0$ and $x = 29$ gives the solution $N = 420$.

*****

GENERATION OF FIBONACCI NUMBERS BY DIGITAL FILTERS

SALAH M. YOUSIF
California State University, Sacramento, California

ABSTRACT

This paper presents some applications of Fibonacci numbers in system and communication theory. Methods of generating Fibonacci sequences and codes by sequential binary filters are given.

INTRODUCTION

The role that Fibonacci numbers play in system theory is worthy of engineering investigations. Fibonacci numbers find their way in algebraic coding theory in communications, linear sequential circuits, and linear digital filters. Although some of these applications are not direct realizations of Fibonacci numbers, they provide the conceptual framework for the related model. For example, the concept of recurrence equation that generates the numbers is utilized to generate difference codes which are used in radar ranging by long-range radars, such as satellite tracking radars and radars that are used for planet's ranging [2]. Another example of Fibonacci numbers is one used to generate a model for population growth in animal and biological colonies. Digital realizations of these models will be given later in the sequel. We will introduce some general applications of Fibonacci numbers and present their digital filter realizations. Then, we present Fibonacci recurrence codes, and give an example of a binary digital sequential circuit to generate these codes.

GENERAL APPLICATIONS

The Z-transforms of discrete time function $y(k)$; $k = 0, 1, 2, \ldots$, is defined by:

(1) $$Z\{y(k)\} = \sum_{k=0}^{\infty} y(k)Z^{-k}.$$

The Z-transform of $y(k + n)$, $n > 0$, is given by:

(2) $$Z\{Y(k + n)\} = Z^n Y(Z) - Z^n \sum_{j=0}^{n-1} Y(j)Z^{-j}.$$

The Z-transform of the Fibonacci equation, after rearrangement,

(3) $$y(k + 1) = y(k) + y(k - 1); \quad y(0) = 0, \quad y(1) = 1,$$

is given by:

(4) $$(Z^2 - Z - 1)Y(Z) = 0$$

The equation

(5) $$Z^2 - Z - 1 = 0$$

is the characteristic equation of the Fibonacci recurrence equation (3). The digital filter that realizes equation (4), and generates the Fibonacci sequence represented by equation (3), is shown in Figure 1.

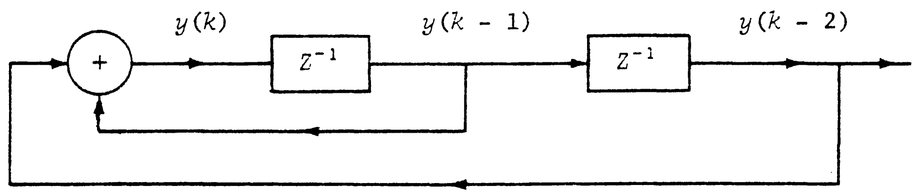

FIGURE 1. Fibonacci Sequence Generator

It is understood that the small box that contains Z^{-1} in Figure 1 represents a unit delay. Unfortunately, the above filter is unstable, since one of the roots has absolute value more than unity. However, this unstable behavior can be of great advantage if we assume that $y(k)$, $k = 0, 1, 2, \ldots$, are elements in a field $GF(p)$ of prime characteristic p. An example of application of the Fibonacci sequences is modeling of population growth of rabbit population by:

(6) $$y(k) = y(k - 1) + y(k - 2) + u(k) \ldots$$

where $y(k)$ represents number of pairs of rabbits at the kth month, and $u(k)$, $k = 0, 1, 2, \ldots$, is a control sequence which, if chosen properly, yields a stable population. The control sequence $u(k)$ may be chosen as feedback linear combination of $y(k - 1)$ and $y(k - 2)$, that is:

(7) $$u(k) = -\beta_1 y(k - 1) - \beta_2 y(k - 2).$$

Substituting (7) in (6) yields the equation:

(8) $$y(k) = (1 - \beta_1)y(k - 1) + (1 - \beta_2)y(k - 2)$$

whose characteristic equation is given by:

(9) $$Z^2 + (\beta_1 - 1)Z + (\beta_2 - 1) = 0.$$

Clearly the roots of the characteristic equation (9) can be assigned arbitrarily by proper choice of β_1 and β_2.

A filter realization of this model is shown in Figure 2. The circles represent multipliers by:

$$\alpha_1 = 1 - \beta_1 \quad \text{and} \quad \alpha_2 = 1 - \beta_2.$$

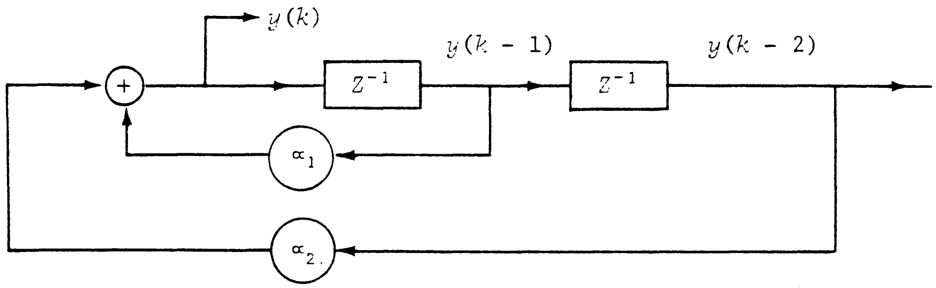

FIGURE 2. Controlled Population Model

DIFFERENCE CODES

Difference code transforms a given m-digit initial sequence $a_0, a_1, \ldots, a_{m-1}$ into an infinitely long sequence $y_0, y_1, y_2, \ldots$, sequentially by the linear difference equation,

$$b_0 y(k) + b_1 y(k-1) + \cdots + b_m y(k-m) = 0;\ b_0,\ b_m \neq 0;$$

$$k = m,\ m-1,\ \ldots,\ a_0 = y(0),\ a_1 = y(1),\ \ldots,\ a_{m-1} = y(m-1),$$

where $y(k)$ and $b(k)$ are elements of the finite Galois field, $GF(p)$. The left characteristic equation of the difference equation (10) is:

(11) $$C(Z) = b_0 + b_1 Z + \cdots + b_m Z^m.$$

The generating function of this code is given by the formal power series $G(Z)$:

(12) $$G(Z) = y(0) + y(1)Z + y(2)Z^2 + \cdots = \sum_{n=0}^{\infty} y(n)Z^n.$$

It can be shown [2] that each solution

$$y(0),\ y(1),\ y(2),\ \ldots,$$

corresponding to (10) has a generating function

(13) $$G(Z) = \sum_{n=0}^{\infty} y(n)Z^n = \frac{A(Z)}{G(Z)},$$

where $A(Z)$ is the polynomial in Z with the initial sequence $a_0, a_1, \ldots, a_{m-1}$ as coefficients; that is,

(14) $$A(Z) = a_0 + a_1 Z + \cdots + a_{m-1} Z^{m-1}.$$

The generation function is obtained from (13) by long division over the specified field. For example, over the field of real numbers, the Fibonacci sequence is given as coefficients of the power series $G(Z)$ given by:

(15) $$G(Z) = \frac{1}{1 - Z - Z^2} = 1 + Z + 2Z^2 + 3Z^3 + 5Z^4 + \cdots.$$

It is not difficult to see that difference codes over finited fields are periodic. The Fibonacci sequence over the binary field has the generating function

(16) $$G(Z) = \frac{A(Z)}{G(Z)} = \frac{1}{1 + Z + Z^2} = 1 + Z + Z^3 + Z^4 + Z^6 + Z^7 + Z^9 + \cdots.$$

$A(Z) = 1 + (0)Z$, since the initial code word is given the initial sequence $a_0 = 1$, $a_1 = 0$. Therefore, the difference code given by (16) is periodic, with period 3, and has the form 110110110110... .

Periodic codes of maximal period are of interest in long-range radar ranging, especially those used in satellite tracking. Those codes are generated by difference equations whose characteristic equations are primitive, with respect to the given finite field. The polynomial $C(Z)$ of degree n is primitive over the field $GF(p)$ if $C(Z)$ divides $Z^{(p^n-1)} - 1$ and it divides no polynomial $(Z^t - 1)$ with $t < p^n - 1$. The difference code whose characteristic polynomial is primitive and has degree n, is maximal period code. The maximal period of the code equals $p^n - 1$, where p is the prime characteristic of the field. Over the binary field $GF(2)$, the primitive characteristic polynomial, $C(Z)$, of the Fibonacci equation is given by:

(17) $$C(Z) = 1 + Z + Z^2.$$

$C(Z)$ is primitive of degree 2. Therefore, the code generated by $C(Z)$ is periodic with maximal period 3. Long period difference codes of this type are usually used in satellite communications. As an example, the primitive polynomial $1 + x + x^{22}$ generates a code sequence of a period $2^{22} - 1 \simeq 4{,}194{,}393$.

The Fibonacci code sequence over $GF(2)$ has correlation function $R(\ell) = -1$ for all shifts ℓ except for $\ell = 0$ and multiples of $2^2 - 1$, at which the value of $R(\ell)$ is $2^2 - 1 = 3$. The correlation property is of great importance in the ranging operation of satellite radars.

It has been shown that Fibonacci sequences can be used in coding and communication theory, and can be implemented by binary digital filters. Similar applications can utilize this approach to generate Fibonacci numbers.

REFERENCES

1. Brother U. Alfred. *An Introduction to Fibonacci Discovery*. San Jose, Calif.: The Fibonacci Association, 1965.
2. Garret Birkhoff and Thomas C. Bartee. *Modern Applied Algebra*. New York: McGraw-Hill Book Company, Inc., 1970.
3. James A. Cadzow. *Discrete-Time Systems*. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1973.

*****

THE FIBONACCI SERIES IN THE DECIMAL EQUIVALENTS OF FRACTIONS

CHARLES F. WINANS
Akron, Ohio

SUMMARY

Four numbers below 100, as denominators of fractions, yield decimal equivalents in which the sequence of digits can also be produced by summations of the terms of the Fibonacci series.

Where every Fibonacci term is used, and moving each term one place to the right, the sequence is that for 1/89; using every second term, the sequence is that for 1/71; with every third term, 2/59; and with every fourth term, 3/31.

The larger denominators: 9899, 9701, 9599, 9301, 8899, 8201, 7099, 6301, and 2399, give repeating decimal equivalents which can be obtained by the summations of every Fibonacci term, every second, third, ..., up to every ninth term, in this case moving each successive term two places to the right. Moreover, the numerators associated with these denominators are: 1, 1, 2, 3, 5, 8, 13, 21, and 34, the first nine terms in the Fibonacci series.

Still larger denominators yield Fibonacci decimal equivalents. Using every fourteenth term, and moving each term three places to the right, the sequence for 377/15701 is obtained.

The decimal equivalents for 9/71, 1/109, 1/10099, and others, can be generated from *right* to *left* by a reverse summation of Fibonacci terms.

The Lucas-, Negative Fibonacci-, Tribonacci-, and other series produce sequences of digits in repeating decimals.

INTRODUCTION

The Fibonacci series is thus defined: $F_1 = 1$; $F_2 = 1$; $F_{(n-1)} + F_n = F_{(n+1)}$; and the first several terms are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... Recently, Brousseau [1] called attention to the fact that the sequence of digits in the decimal equivalent of 1/89 is developed by a summation of the Fibonacci series where each successive term is moved one place to the right; thus,

```
112358
     13
      21
       34
        55
         89
         144
           233
            377
            ...
-----------------
11235955056 ...
```