Partial Solutions

Section 2

1–2. See list on page 83. **3.** Several will be developed later in the text.
4. 1, 4, 5, 9, 14, . . . , 7375, 11,933, 19,308

Section 3

1. Using the quadratic formula with $a = 1$, $b = -1$, and $c = -1$, you have
$$x = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2};\ \alpha = \frac{1 + \sqrt{5}}{2},\ \beta = \frac{1 - \sqrt{5}}{2}.$$

2. $\alpha \doteq \dfrac{1 + 2.236}{2} \doteq 1.618;\ \beta \doteq \dfrac{1 - 2.236}{2} \doteq -.618$

3. a. $\alpha + \beta = \dfrac{1 + \sqrt{5}}{2} + \dfrac{1 - \sqrt{5}}{2} = 1$

b. $\alpha - \beta = \dfrac{1 + \sqrt{5}}{2} - \dfrac{1 - \sqrt{5}}{2} = \sqrt{5}$

c. $\alpha\beta = \left(\dfrac{1 + \sqrt{5}}{2}\right)\left(\dfrac{1 - \sqrt{5}}{2}\right) = \dfrac{(1)^2 - (\sqrt{5})^2}{4} = \dfrac{1 - 5}{4} = -1$

4. $$1 + \frac{1}{\alpha} = 1 + \frac{1}{\dfrac{1 + \sqrt{5}}{2}} = 1 + \frac{2(1 - \sqrt{5})}{(1 + \sqrt{5})(1 - \sqrt{5})} = \frac{1 + \sqrt{5}}{2} = \alpha$$

5. a. $\alpha^3 = \alpha(\alpha^2) = \alpha(\alpha + 1) = \alpha^2 + \alpha = (\alpha + 1) + \alpha = 2\alpha + 1$
b. $\alpha^4 = \alpha(\alpha^3) = \alpha(2\alpha + 1) = 2\alpha^2 + \alpha = 2(\alpha + 1) + \alpha = 3\alpha + 2$
c. $\alpha^5 = \alpha(\alpha^4) = \alpha(3\alpha + 2) = 3\alpha^2 + 2\alpha = 3(\alpha + 1) + 2\alpha = 5\alpha + 3$
Note: It can be proved by mathematical induction (reviewed on page 54) that, in general, $\alpha^n = F_n\alpha + F_{n-1}$, $n \geq 1$.

6. $$\alpha^3 - \frac{1}{\alpha^3} = 2\alpha + 1 - \frac{1}{2\alpha + 1} = 2 + \sqrt{5} - \frac{1}{2 + \sqrt{5}}\left(\frac{2 - \sqrt{5}}{2 - \sqrt{5}}\right)$$
$$= 2 + \sqrt{5} + 2 - \sqrt{5} = 4 = L_3$$

7. Since $\alpha\beta = -1$, $\alpha^{-1} = -\beta$; $\dfrac{\alpha^4 - \alpha^{-4}}{\sqrt{5}} = \dfrac{\alpha^4 - \beta^4}{\alpha - \beta} = F_4$.

Section 4

1. In any triangle, we must have $a + b > c$, $b + c > a$, $c + a > b$. For any three consecutive Fibonacci numbers, $F_n + F_{n+1} = F_{n+2}$, and so there can be no triangle with sides having measures F_n, F_{n+1}, F_{n+2}. In general, consider Fibonacci numbers F_p, F_q, F_r, where $F_p \leq F_{q-1}$ and $F_{q+1} \leq F_r$. Since $F_{q-1} + F_q = F_{q+1}$ and $F_p \leq F_{q-1}$, we have $F_p + F_q \leq F_{q+1}$, and since $F_{q+1} \leq F_r$, we have $F_p + F_q \leq F_r$. Therefore, there can be no triangle with sides having measures F_p, F_q, and F_r.

2. The measures of the sides of the triangles must be (page 17) a, ra, r^2a and ra, r^2a, r^3a with $r \neq 1$ (page 21). Therefore, it is impossible to have $a = ra$ or $a = r^2a$ or $ra = r^2a$, and so on, and so neither triangle can be isosceles.

3. a. $y = (x - a)(x - b),\ a < b$
If $a < x < b$, then $x - a > 0$ and $x - b < 0$, and so $y < 0$.
If $x \leq a$, then $x - a \leq 0$ and $x - b < 0$, and so $y \geq 0$.
If $x \geq b$, then $x - a > 0$ and $x - b \geq 0$, and so $y \geq 0$.

b. If $a = b$, then $y = (x - a)^2$; $(x - a)^2 \geq 0$ for all x.

4. a. $r^2 - r - 1 = 0$ has roots α and β, $\beta < \alpha$. Thus, $r^2 - r - 1 = (r - \alpha)(r - \beta)$, and from Ex. 3a, $r^2 - r - 1 < 0$ for $\beta < r < \alpha$.

b. $r^2 + r - 1 = 0$ has roots $-\alpha$ and $-\beta$, $-\alpha < -\beta$. Thus, $r^2 + r - 1 = [r - (-\alpha)][r - (-\beta)]$, and from Ex. 3a, $r^2 + r - 1 > 0$ when $r < -\alpha$ (both factors negative) or $r > -\beta$ (both factors positive). (Of course, in the application in Prob. 4, $r > 0$.)

c. Since $-\beta = \dfrac{\sqrt{5} - 1}{2}$ and $\alpha = \dfrac{1 + \sqrt{5}}{2}$, $-\beta < \alpha$. Since $\beta < -\beta < \alpha$,

$$\{r\colon \beta < r < \alpha\} \cap \{r\colon r < -\alpha \ \text{ or } \ r > -\beta\} = \{r\colon -\beta < r < \alpha\}.$$

Since $\alpha\beta = -1$, we have $-\beta = \dfrac{1}{\alpha}$, and $\dfrac{1}{\alpha} < r < \alpha$.

5. From Ex. 3b, $(r - 1)^2 \geq 0$ for all r. Thus, $r^2 - 2r + 1 \geq 0$, or $r^2 + 1 \geq 2r$, for all r. If $r > 0$, $2r > r$, and so $r^2 + 1 > r$ for $r > 0$.

6. If $r > 1$, $a < ar < ar^2$ and $\dfrac{a}{r^2} < \dfrac{a}{r} < a$. The sides can be paired by measures:

$$a \leftrightarrow \frac{a}{r^2},\quad ar \leftrightarrow \frac{a}{r},\quad ar^2 \leftrightarrow a.\quad \frac{\frac{a}{r^2}}{a} = \frac{\frac{a}{r}}{ar} = \frac{a}{ar^2} = \frac{1}{r^2},\ \text{the ratio of similarity.}$$

7. Since $CG = DB = s + t$ and $CD = GB = s$, $\dfrac{CG}{GB} = \dfrac{s + t}{s} = \dfrac{s}{t} = \alpha$, and so $DCGB$ is a Golden Rectangle.

8. Since $\triangle AFB \sim \triangle FEB$, $\dfrac{AF}{FB} = \dfrac{FE}{EB} = \dfrac{s}{t} = \alpha$.

9. a. Since $\dfrac{s}{t} = \alpha$, $\dfrac{s^2 + 2st}{s^2 + t^2} = \dfrac{\dfrac{s^2}{t^2} + 2\left(\dfrac{s}{t}\right)}{\dfrac{s^2}{t^2} + 1} = \dfrac{\alpha^2 + 2\alpha}{\alpha^2 + 1} = \dfrac{\alpha(\alpha + 2)}{\alpha + 2} = \alpha$.

b. $\dfrac{t^2 - 2st}{s^2 + t^2} = \dfrac{s^2 + t^2}{s^2 + t^2} - \dfrac{s^2 + 2st}{s^2 + t^2} = 1 - \alpha = \beta$

c. $\dfrac{s^2 + 4st - t^2}{s^2 + t^2} = \dfrac{s^2 + 2st}{s^2 + t^2} - \dfrac{t^2 - 2st}{s^2 + t^2} = \alpha - \beta = \sqrt{5}$

Section 5

1. $F_{2n} = \dfrac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = \left(\dfrac{\alpha^n - \beta^n}{\alpha - \beta}\right)(\alpha^n + \beta^n) = F_n L_n,\ n \geq 1$

2.
$$F_{n-1} + F_{n+1} = \frac{(\alpha^{n-1} - \beta^{n-1}) + (\alpha^{n+1} - \beta^{n+1})}{\alpha - \beta}$$
$$= \frac{\alpha^{n+1} + \alpha^{n-1} - \beta^{n+1} - \beta^{n-1}}{\alpha - \beta} = \frac{\alpha^{n+1} - (\alpha\beta)\alpha^{n-1} + (\alpha\beta)\beta^{n-1} - \beta^{n+1}}{\alpha - \beta}$$
$$= \frac{\alpha^n(\alpha - \beta) + \beta^n(\alpha - \beta)}{\alpha - \beta} = \alpha^n + \beta^n = L_n,\ \text{since } \alpha\beta = -1.$$

3. $L_{-n} = \alpha^{-n} + \beta^{-n}$, $n > 0$. Since $\alpha\beta = -1$, we have $\alpha^{-n} = (-1)^n\beta^n$ and $\beta^{-n} = (-1)^n\alpha^n$. Thus, $L_{-n} = (-1)^n(\alpha^n + \beta^n) = (-1)^n L_n$.

4. $F_n^2 = \left(\dfrac{\alpha^n - \beta^n}{\sqrt{5}}\right)^2$. Thus, $5F_n^2 = (\alpha^n - \beta^n)^2 = \alpha^{2n} - 2\alpha^n\beta^n + \beta^{2n}$
$= \alpha^{2n} + \beta^{2n} - 2(-1)^n = L_{2n} - 2(-1)^n$.

5. $L_n^2 = (\alpha^n + \beta^n)^2 = \alpha^{2n} + 2\alpha^n\beta^n + \beta^{2n} = \alpha^{2n} + \beta^{2n} + 2(-1)^n = L_{2n} + 2(-1)^n$.

6. $F_{n+1}L_n - L_{n+1}F_n$

$$= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}(\alpha^n + \beta^n) - (\alpha^{n+1} + \beta^{n+1})\frac{\alpha^n - \beta^n}{\alpha - \beta}$$

$$= \frac{\alpha^{2n+1} - \beta^{n+1}\alpha^n + \alpha^{n+1}\beta^n - \beta^{2n+1} - \alpha^{2n+1} - \beta^{n+1}\alpha^n + \alpha^{n+1}\beta^n + \beta^{2n+1}}{\alpha - \beta}$$

$$= \frac{-\beta^{n+1}\alpha^n + \alpha^{n+1}\beta^n - \beta^{n+1}\alpha^n + \alpha^{n+1}\beta^n}{\alpha - \beta} = \frac{2(\alpha^{n+1}\beta^n - \alpha^n\beta^{n+1})}{\alpha - \beta}$$

$$= \frac{2(\alpha^n\beta^n)(\alpha - \beta)}{\alpha - \beta} = 2(-1)^n$$

7. $\dfrac{F_{n+1}}{F_n} - \dfrac{L_{n+1}}{L_n} = \dfrac{F_{n+1}L_n - L_{n+1}F_n}{F_nL_n} = \dfrac{2(-1)^n}{F_{2n}}$ by Ex. 6 and 1.

8. If $n < 0$, then $-n > 0$. For $-n > 0$, $F_{-(-n)} = (-1)^{-n+1}F_{-n}$,

or $F_{-n} = \dfrac{F_n}{(-1)^{-n+1}} = \dfrac{(-1)^{n+1}F_n}{(-1)^2} = (-1)^{n+1}F_n$.

9. If $n < 0$, then $-n > 0$. For $-n > 0$, $L_{-(-n)} = (-1)^{-n}L_{-n}$,

or $L_{-n} = \dfrac{L_n}{(-1)^{-n}} = (-1)^n L_n$.

10. From Ex. 4, $5F_n^2 = L_{2n} - 2(-1)^n$. From Ex. 5, $L_{2n} = L_n^2 - 2(-1)^n$. Thus, $5F_n^2 = L_n^2 - 4(-1)^n$.

11. $L_n = F_{n+1} + F_{n-1} = (F_{n+2} - F_n) + (F_n - F_{n-2}) = F_{n+2} - F_{n-2}$

Section 6

1. $F_{12} \doteq \dfrac{\alpha^{12}}{\sqrt{5}}$. $\log \dfrac{\alpha^{12}}{\sqrt{5}} \doteq 12(0.20898) - 0.3494 \doteq 2.5078 - 0.3494 \doteq 2.1584$.

Thus, $F_{12} = 144$.

2. $L_{12} \doteq \alpha^{12}$. $\log \alpha^{12} \doteq 2.5078$. Thus, $L_{12} = 322$.

3. $F_{34} \doteq \dfrac{\alpha^{34}}{\sqrt{5}} \doteq 570 \times 10^4$

4. $L_{33} \doteq \alpha^{34} \doteq 788 \times 10^4$

5. If $F_n = 1597$, then $F_{n+1} = \left[\dfrac{1597 + 1 + \sqrt{12752045}}{2}\right] = \left[\dfrac{1598 + 3571}{2}\right]$

$$= \left[\frac{5169}{2}\right] = [2584.5] = 2584.$$

6. If $L_n = 2207$, then $L_{n+1} = \left[\dfrac{2207 + 1 + \sqrt{24354245}}{2}\right] = \left[\dfrac{2208 + 4935}{2}\right]$

$$= \left[\frac{7143}{2}\right] = [3571.5] = 3571.$$

Section 7

1. $F_7 = 13$; $F_{14} = 13(29)$; $F_{21} = 13(842)$; $F_{28} = 13(24{,}447)$
2. $F_{10} = 55$; $F_{20} = 55(123)$; $F_{30} = 55(15{,}128)$; $F_{40} = 55(1{,}860{,}621)$
3. $F_{24} = 46{,}368 = 2(23{,}184) = 3(15{,}456) = 8(5796) = 21(2208) = 144(322)$
4. $F_{30} = 832{,}040 = 2(416{,}020) = 5(166{,}408) = 8(104{,}005)$
   $= 55(15{,}128) = 610(1364)$
5. $L_4 = 7$; $F_8 = 7(3)$; $F_{16} = 7(141)$
6. $L_7 = 29$; $F_{14} = 29(13)$; $F_{28} = 29(10{,}959)$
7. $L_4 = 7$; $L_{12} = 7(46)$; $L_{20} = 7(2161)$
8. $L_5 = 11$; $L_{15} = 11(124)$; $L_{25} = 11(15{,}251)$
9. $F_{12} = 144 = 2(76 - 4) = 2(4)(18) = 3(47 + 1) = 8(18)$
10. $F_{18} = 2584 = 2(1364 - 76 + 4) = 8(322 + 1) = 34(76)$
11. $(F_{16}, F_{24}) = F_{(16,\,24)} = F_8 = 21$
12. $(F_{24}, F_{36}) = F_{(24,\,36)} = F_{12} = 144$

Section 8

1. Since 13 is F_7, the entry point is 7. The remainders on dividing by 13 are $R_0 = 0$, $R_1 = 1, \ldots,$ $R_7 = 0$, $R_8 = 8, \ldots,$ $R_{14} = 0$, $R_{15} = 12, \ldots,$ $R_{21} = 0$, $R_{22} = 5, \ldots,$ $R_{28} = 0$, $R_{29} = 1, \ldots,$ and so $K_{13} = 28$.
2. The remainders on dividing by 13 are $R_0 = 2$, $R_1 = 1, \ldots,$ $R_{19} = 2$, $R_{20} = 8, \ldots,$ $R_{23} = 2$, $R_{24} = 7, \ldots,$ $R_{28} = 2$, $R_{29} = 1, \ldots,$ and so $K_{13} = 28$. No remainder is 0, and so 13 does not divide any Lucas number.
3. The remainders on dividing by 10 are $R_0 = 0$, $R_1 = 1, \ldots,$ $R_{15} = 0$, $R_{16} = 7, \ldots,$ $R_{30} = 0$, $R_{31} = 9, \ldots,$ $R_{45} = 0$, $R_{46} = 3, \ldots,$ $R_{60} = 0$, $R_{61} = 1, \ldots,$ and so $K_{10} = 60$. The entry point is 15.
4. The remainders on dividing by 6 are $R_0 = 2$, $R_1 = 1, \ldots,$ $R_6 = 0, \ldots,$ $R_{15} = 2$, $R_{16} = 5, \ldots,$ $R_{18} = 0, \ldots,$ $R_{21} = 2$, $R_{22} = 3, \ldots,$ $R_{24} = 2$, $R_{25} = 1, \ldots,$ and so $k_6 = 24$. The entry point is 6.
5. The remainders on dividing by 10 are $R_0 = 2$, $R_1 = 1, \ldots,$ $R_{12} = 2$, $R_{13} = 1, \ldots,$ and so $K_{10} = 12$. No remainder is 0, and so 10 has no entry point in the Lucas numbers. (Also, since no Lucas number is divisible by 5 (page 45), no Lucas number is divisible by 10.)

Section 9

From (A) on page 49:

1. With $x = 1$ and $y = x$, $$\sum_{i=0}^{n} \binom{n}{i} x^i = (1 + x)^n.$$
2. With $x = 1, y = 1$, $$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$
3. With $x = 1, y = -1$, $$\sum_{i=0}^{n} \binom{n}{i} (-1)^i = 0.$$
4. $$\binom{n}{0} = \frac{n!}{0!(n-0)!} = 1; \quad \binom{n}{n} = \frac{n!}{n!(n-n)!} = 1. \quad \binom{n}{m} = \frac{n!}{m!(n-m)!};$$
$$\binom{n-1}{m} + \binom{n-1}{m-1} = \frac{(n-1)!}{m!(n-1-m)!} + \frac{(n-1)!}{(m-1)!(n-m)!}$$
$$= \frac{(n-m)(n-1)! + m(n-1)!}{m!(n-m)!} = \frac{n!}{m!(n-m)!} = \binom{n}{m}$$
5. $$\sum_{i=0}^{4} \binom{4}{i} F_i = \binom{4}{0} F_0 + \binom{4}{1} F_1 + \binom{4}{2} F_2 + \binom{4}{3} F_3 + \binom{4}{4} F_4$$
$$= F_0 + 4F_1 + 6F_2 + 4F_3 + F_4 = 21 = F_8$$

6. $$\sum_{i=0}^{n}\binom{n}{i}F_{i+j} = \frac{1}{\alpha-\beta}\left[\sum_{i=0}^{n}\binom{n}{i}\alpha^{i+j} - \sum_{i=0}^{n}\binom{n}{i}\beta^{i+j}\right]$$
$$= \frac{1}{\alpha-\beta}\left[\alpha^{j}(1+\alpha)^{n} - \beta^{j}(1+\beta)^{n}\right] = \frac{1}{\alpha-\beta}[\alpha^{2n+j} - \beta^{2n+j}]$$
$= F_{2n+j}$, since $1+\alpha = \alpha^2$ and $1+\beta = \beta^2$.

7. $$\sum_{i=0}^{n}\binom{n}{i}L_{i+j} = \sum_{i=0}^{n}\binom{n}{i}\alpha^{i+j} + \sum_{i=0}^{n}\binom{n}{i}\beta^{i+j}$$
$$= \alpha^{j}(1+\alpha)^{n} + \beta^{j}(1+\beta)^{n} = \alpha^{2n+j} + \beta^{2n+j} = L_{2n+j}$$

8. $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}F_{i+j} = \frac{1}{\alpha-\beta}\left[\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\alpha^{i+j} - \sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\beta^{i+j}\right]$$
$$= \frac{1}{\alpha-\beta}\left[\alpha^{j}\sum_{i=0}^{n}\binom{n}{i}(-\alpha)^{i} - \beta^{j}\sum_{i=0}^{n}\binom{n}{i}(-\beta)^{i}\right]$$
$$= \frac{1}{\alpha-\beta}[\alpha^{j}(1-\alpha)^{n} - \beta^{j}(1-\beta)^{n}] = \frac{\alpha^{j}\beta^{n} - \beta^{j}\alpha^{n}}{\alpha-\beta}$$
$$= \frac{(\alpha\beta)^{j}(\beta^{n-j} - \alpha^{n-j})}{\alpha-\beta} = (-1)^{j+1}F_{n-j}$$

9. $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}F_{2i+j} = \frac{1}{\alpha-\beta}\left[\alpha^{j}\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\alpha^{2i} - \beta^{j}\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\beta^{2i}\right].$$
But $(-1^{i})(\alpha^2)^{i} = (-\alpha^2)^{i}$ and $(-1^{i})(\beta^2)^{i} = (-\beta^2)^{i}$.
Thus, $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}F_{2i+j} = \frac{1}{\alpha-\beta}[\alpha^{j}(1-\alpha^2)^{n} - \beta^{j}(1-\beta^2)^{n}]$$
$$= \frac{1}{\alpha-\beta}[\alpha^{j}(-\alpha)^{n} - \beta^{j}(-\beta)^{n}] = (-1)^{n}\left(\frac{\alpha^{n+j} - \beta^{n+j}}{\alpha-\beta}\right) = (-1)^{n}F_{n+j}\,.$$

Section 10

1. $\sum_{i=1}^{n} F_i = F_{n+2} - 1,\ n \geq 1$

3. $\sum_{i=1}^{n} F_i^2 = F_nF_{n+1},\ n \geq 1$

4. $F_{n+1} = F_n + F_{n-1};\ F_{n+1}^2 = F_n^2 + 2F_nF_{n-1} + F_{n-1}^2.$
From (I_3), $F_nF_{n-1} = F_1^2 + \cdots + F_{n-1}^2.$
Thus, $F_{n+1}^2 = F_n^2 + 3F_{n-1}^2 + 2(F_{n-2}^2 + \cdots + F_1^2).$
The number of squares is $1 + 3 + 2(n-2) = 2n$.

7. $\sum_{i=1}^{n} F_{2i-1} = F_{2n},\ n \geq 1$

9. $\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1,\ n \geq 1$

10. $L_1 = L_2 - L_0$
$L_3 = L_4 - L_2$
$L_5 = L_6 - L_4$
. . . . . .
$L_{2n-3} = L_{2n-2} - L_{2n-4}$
$L_{2n-1} = L_{2n} - L_{2n-2}$
$$\sum_{i=1}^{n} L_{2i-1} = L_{2n} - 2$$

11. $L_2 = L_3 - L_1$
$L_4 = L_5 - L_3$
$L_6 = L_7 - L_5$
. . . . . .
$L_{2n-2} = L_{2n-1} - L_{2n-3}$
$L_{2n} = L_{2n+1} - L_{2n-1}$
$$\sum_{i=1}^{n} L_{2i} = L_{2n+1} - 1$$

12. $L_n = F_{n-1} + F_{n+1}$ (I8); $L_{n-1} = F_{n-2} + F_n$; $L_{n+1} = F_n + F_{n+2}$.

$$\tfrac{1}{5}(L_{n-1} + L_{n+1}) = \tfrac{1}{5}(F_{n-2} + 2F_n + F_{n+2})$$
$$= \tfrac{1}{5}[(F_n - F_{n-1}) + 2F_n + (F_n + F_{n+1})]$$
$$= \tfrac{1}{5}(4F_n + F_n) = F_n$$

13. $F_{2n} = F_nL_n$ (I7); $L_n = F_{n-1} + F_{n+1}$ (I8).

$$F_{2n} = F_n(F_{n-1} + F_{n+1}) = (F_{n+1} - F_{n-1})(F_{n+1} + F_{n-1})$$
$$= F_{n+1}^2 - F_{n-1}^2$$

14. $F_{2n} = F_{n+1}^2 - F_{n-1}^2$ (I10).

$$F_{2n+1} = F_{2(n+1)} - F_{2n} = F_{n+2}^2 - F_n^2 - F_{n+1}^2 + F_{n-1}^2$$
$$= (F_{n+1} + F_n)^2 - F_n^2 - F_{n+1}^2 + (F_{n+1} - F_n)^2 = F_{n+1}^2 + F_n^2$$

15. $h = \dfrac{1}{\sqrt{8^2 + 3^2}} < \dfrac{1}{\sqrt{8^2}}$, or $\dfrac{1}{8}$, since $8^2 + 3^2 > 8^2$.

17. To prove: $H_{n+2} = qF_{n+1} + pF_n$

$H_1 = p$; $H_2 = q$; $H_{n+2} = H_{n+1} + H_n$, $n \geq 1$

$n = 1$: $H_3 = H_2 + H_1 = q + p = qF_2 + pF_1$

$n = 2$: $H_4 = H_3 + H_2 = qF_2 + pF_1 + q = qF_3 + pF_2$

Thus, we have a basis for induction.

We assume (inductive hypothesis): $P(k-1)$: $H_{k+1} = qF_k + pF_{k-1}$

and $P(k)$: $H_{k+2} = qF_{k+1} + pF_k$

Adding, we have: $P(k+1)$: $H_{k+3} = qF_{k+2} + pF_{k+1}$

The proof is complete by mathematical induction.

Note: Observe the variation from the usual pattern of mathematical induction. The inductive basis has two (consecutive) validations, and the inductive hypothesis has two (consecutive) assumptions.

18. $F_{(k+1)n} = F_{kn}F_{n+1} + F_{kn-1}F_n$

To prove: F_n divides F_{nk}, $k > 0$.

$k = 1$: F_n divides F_n. Thus, we have a basis for induction.

We assume (inductive hypothesis): $P(p)$: F_n divides F_{np}.

But $F_{(p+1)n} = (F_{pn})F_{n+1} + F_{pn-1}(F_n)$.

Since F_n divides F_n and is assumed to divide F_{pn}, F_n must divide $F_{(p+1)n}$, and the proof is complete by mathematical induction.

Section 11

1. $A + Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a+0 & b+0 \\ c+0 & d+0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$

$Z + A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0+a & 0+b \\ 0+c & 0+d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$

2. $A + B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$

$= \begin{pmatrix} e+a & f+b \\ g+c & h+d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = B + A$

3. $(A + B) + C = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right] + \begin{pmatrix} i & j \\ k & l \end{pmatrix}$

$= \begin{pmatrix} (a+e)+i & (e+f)+j \\ (c+g)+k & (d+h)+l \end{pmatrix} = \begin{pmatrix} a+(e+i) & e+(f+j) \\ c+(g+k) & d+(h+l) \end{pmatrix}$

$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \left[\begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} i & j \\ k & l \end{pmatrix}\right] = A + (B + C)$

4. $A + (-A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} = \begin{pmatrix} a-a & b-b \\ c-c & d-d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = Z$

5. $(sr)A = \begin{pmatrix} (sr)a & (sr)b \\ (sr)c & (sr)d \end{pmatrix} = \begin{pmatrix} s(ra) & s(rb) \\ s(rc) & s(rd) \end{pmatrix} = s(rA)$

6. $-A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} = \begin{pmatrix} (-1)a & (-1)b \\ (-1)c & (-1)d \end{pmatrix} = (-1)(A)$

7. $AI = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a(1)+b(0) & a(0)+b(1) \\ c(1)+d(0) & c(0)+d(1) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$

$IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1(a)+0(c) & 1(b)+0(d) \\ 0(a)+1(c) & 0(b)+1(d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$

8. $A(B+C) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e+i & f+j \\ g+k & h+l \end{pmatrix}$

$= \begin{pmatrix} a(e+i)+b(g+k) & a(f+j)+b(h+l) \\ c(e+i)+d(g+k) & c(f+j)+d(h+l) \end{pmatrix}$

$= \begin{pmatrix} (ae+bg)+(ai+bk) & (af+bh)+(aj+bl) \\ (ce+dg)+(ci+dk) & (cf+dh)+(cj+dl) \end{pmatrix} = AB + AC$

Similarly, $(B+C)A = BA + CA$.

9. To prove: $\det(Q^n) = (-1)^n, n \geq 1$.
$\det Q = -1$; $\det Q^2 = \det((Q)(Q)) = \det Q \det Q = (-1)^2$
Thus, we have a basis for induction.
We assume (inductive hypothesis): $P(k)$: $\det Q^k = (-1)^k$
We wish to prove $P(k+1)$: $\det Q^{k+1} = (-1)^{k+1}$.
$\det Q^{k+1} = \det((Q)(Q^k)) = \det Q \det Q^k = (-1)(-1)^k = (-1)^{k+1}$
The proof is complete by mathematical induction.

10. $Q^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$; $Q + I = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = Q^2$

Section 12

1. $1 = F_2$
$2 = F_3$
$3 = F_4$
$4 = F_4 + F_2$
$5 = F_4 + F_3$
$6 = F_4 + F_3 + F_2$
$7 = F_4 + F_3 + F_2 + F_1$
$8 = F_6$
$9 = F_6 + F_2$
$10 = F_6 + F_3$
$11 = F_6 + F_4$
$12 = F_6 + F_4 + F_2$
$13 = F_7$
$14 = F_7 + F_2$
$15 = F_7 + F_3$
$16 = F_7 + F_4$
$17 = F_7 + F_4 + F_2$
$18 = F_7 + F_4 + F_3$
$19 = F_7 + F_4 + F_3 + F_2$
$20 = F_7 + F_4 + F_3 + F_2 + F_1$

2. $1 = F_2$
$2 = F_3$
$3 = F_3 + F_2$
$4 = F_3 + F_2 + F_1$
$5 = F_5$
$6 = F_5 + F_2$
$7 = F_5 + F_3$
$8 = F_5 + F_3 + F_1$
$9 = F_5 + F_3 + F_2 + F_1$
There is no possible representation for 10 with F_4 and F_6 missing. The next available one is $F_7 = 13$, which is too large, and 9 is the sum of all the smaller available Fibonacci numbers.

3. There is only one representation of 27 using distinct Fibonacci numbers: $27 = 21 + 5 + 1$. This may be expressed as $27 = F_8 + F_5 + F_2$.

4. $1966 = 1597 + 233 + 89 + 34 + 13 = F_{17} + F_{13} + F_{11} + F_9 + F_7$

5. $32 = 21 + 8 + 3 = F_8 + F_6 + F_4$ (minimal, or Zeckendorf)
$= 13 + 8 + 5 + 3 + 2 + 1 = F_7 + F_6 + F_5 + F_4 + F_3 + F_2$ (max.)

6. $32 = 29 + 3 = L_7 + L_2$ (minimal)
$= 18 + 7 + 4 + 2 + 1 = L_6 + L_4 + L_3 + L_0 + L_1$ (maximal)