APPENDIX

ADDITIONAL REFERENCES AND A BRIEF SURVEY OF RECENT RESULTS

This appendix, added in translation, gives an additional one hundred and twenty-six references relevant to the continuing development of the properties and applications of the Pascal triangle and its generalizations. The first eighty items (Additional References I) have appeared in the literature in the last five years; preceding the list, there is also (below) a brief review of some of these and other results. The remaining items (Additional References II) were published prior to 1987 but were not originally included in the main body of references. All this material is taken from part of the author's recent paper (in Russian): "On the Pascal triangle and its generalizations. The distribution of the binomial and trinomial coefficients modulo 4 in the Pascal triangle and Pascal pyramid," Voprosy Vychisl. i Prikl. Mat., Vyp. 93. Tashkent, 1992, pp. 5-24.

The publications [1-80] from 1987-1991 discuss a variety of complicated problems connected with the further study of the Pascal triangle and its generalizations. Foremost among these are problems related to the divisibility and the distribution with respect to prime or composite moduli of the binomials and other combinatorial numbers appearing in their corresponding arithmetic triangles [2-9, 13, 14, 17, 23, 31-33, 39-51, 56-59, 62, 69-71, 73-76, 78-80]. Other main themes include the construction of fractal arithmetic structures [3, 25, 40, 42, 52-55, 60-63, 66, 67], various generalized Pascal triangles and their properties [11, 12, 15, 18, 20, 26-30, 35-38], and the construction and study of graph models, determinants, and criteria for primality [10, 16, 19, 21, 22, 34, 64, 65, 68, 72, 77]. Many of the less recent publications [81-126] also deal with these problems, as the article titles indicate.

One of the difficult problems connected with the Pascal triangle and its generalizations is that of determining the distribution of the binomial coefficients, and other combinatorial numbers occurring in arithmetic structures, with respect to a prime modulus, and especially with respect to a composite modulus. Analytic solutions of this problem are discussed in [1-8, 23, 45, 73-76, 78, 94, 95, 110].

The problem of the distribution of the binomial coefficients with respect to a prime modulus p was first discussed in 1957 by J.B. Roberts [110]. He gave, for p=3,5, exact formulas for the number of binomial coefficients congruent modulo p in the Pascal triangle whose base is the row numbered $N=p^n-1$.

In 1978 E. Hexel and H. Sachs [95] generalized the results in [110] by devising a method for determining the number of congruent binomial coefficients not just in the triangle as a whole, but in an arbitrary one of its rows. Again for p=3,5, they established exact formulas for any row of the Pascal triangle with base $N=p^n-1$.

In [1-5], the present author employed normalized p-Latin squares and their associated matrices to derive exact formulas for the distributions of binomial and trinomial coefficients, Stirling numbers of the first and second kinds, Gaussian binomial coefficients, and Euler numbers, in their corresponding arithmetic structures, with respect to the moduli p=3,5,7.

M. Sved [73-76], using a computer, implemented the construction of the triangular distributions of the binomial and Gaussian binomial coefficients, Stirling numbers of the first and second kinds, and Euler and other numbers, with respect to prime and composite moduli for a considerable number of rows of the corresponding triangles.

The references mentioned above are essentially all concerned with the distributions of coefficients and combinatorial numbers with respect to a prime modulus. Only in 1991 have the first results begun to appear for a composite modulus d: K. Davis and W. Webb [23] established an exact formula for the distribution of the binomial coefficients mod 4 in an arbitrary row of the Pascal triangle. F. Howard [45] then generalized the results of [23] and [95] to the case of Gaussian binomial and multinomial coefficients. And even more recently the present author, in the paper mentioned in the introductory paragraph, has obtained exact formulas for the distributions of the binomial and trinomial coefficients modulo 4.

ADDITIONAL REFERENCES

I

1. Bondarenko, B.A. Obobshchennye Treugol'niki i Piramidy Paskalya, ikh Fraktali, Grafy, i Prilozheniya. - Tashkent: FAN, 1990. 192 S.

2. Bondarenko, B.A. Normalizovannye p-Latinskie kvadraty i svyazannye s nimi rekurrentnye uravneniya // Dokl. AN UzSSR. - 1991 - No. 8 - S. 4-6.

3. Bondarenko, B.A. Diskretnye modeli "Kovra Serpinskogo" i ego obobshchenii // Izv. AN UzSSR. Ser. Tekhn. Nauk. - 1991. - No. 4 - S. 3-9.

4. Bondarenko, B.A. Normalizovannye p-Latinskie kvadraty i algoritmy ikh primeneniya k zadacham raspredeleniya elementov po modulyu p v arifmeticheskikh strukturakh // Vopr. Vychisl. i Prikl. Matematiki. - Tashkent: NISO AN UzSSR. - 1991 - Vyp. 91 - S. 61-80.

5. Bondarenko, B.A. Raspredelenie kombinatornykh chisel po modulyu p v arifmeticheskikh treugol'nikakh // Vopr. Vychisl. i Prikl. Matematiki. - Tashkent: NISO AN UzSSR. - 1991. - Vyp. 92. - S. 17-36.

6. Volodin, N.A. O raspredelenii polinomial'nykh koeffitsientov, sravnimykh po mod p^n // Mat. Zametki. - 1989. - T. 45, No. 3. - S. 24-29.

7. Volchegurskii, V.I.; Kerov S.B. Af- algebry usechennykh treugol'nikov Paskalya // Vopr. Teorii Grupi i Gomologichnoi Algebry. - Yaroslavl', 1989. - S. 29-38.

8. Karachik, V.V. O nekotorykh svoistvakh p-Latinskikh matrits i ikh primennii // Vopr. Vychisl. i Prikl. Matematiki. - Tashkent: NISO AN RUz. - 1992. - Vyp. 93. - S. 49-60.

9. Trakhtman, Yu. A. Delimost' raznostei, sostavlennykh iz binomial'nykh koeffitsientov // Issled. po Teorii Chisel. - Saratov, 1987. - No. 9. - S. 131-137.

10. Adams W., Liverance E., Shanks D. Infinitely many necessary and sufficient conditions for primality // Bull. Inst. Combinatorics Appl. - 1991. - V. 3. - P. 69-76.

11. Arkin J., Arney D.C., Giordano F.R. The original manuscript of the generalized Fibonacci numbers combined with the generalized Pascal triangle // Abstracts of papers presented to AMS. - 1990. - V. 11, N 4. - P. 308.

12. Ayoub A.B. The harmonic triangle and the beta function // Math. Mag. - 1987. - V. 60, N 4. - P. 223-225.

13. Bailey D.F. Two-p^3 variations of Lucas theorem // J. Number Theory. - 1990. - V. 35, N 2. - P. 208-215.

14. Bailey D.F. Some binomial coefficient congruences // Appl. Math. Lett. - 1991. - V. 4, N 4. - P. 1-5.

15. Bicknell-Johnson M. Retrograde renegades and Pascal connection: repeating decimals represented by Fibonacci and other sequences appearing from right to left // Fibonacci Quart. - 1989. - V. 27, N 5. - P. 448-457.

16. Bollinger R.C. The Mann-Shanks primality criterion in the Pascal-T triangle T_3 // Fibonacci Quart. - 1989. - V. 27, N 3. - P. 272-275.

17. Bollinger R.C., Burchard C.L. Lucas's theorem and some related results for extended Pascal triangles // Amer. Math. Monthly. - 1990. - V. 96, N 9. - P. 72-78.

18. Cakič N. A note on Euler's numbers // Fibonacci Quart. - 1991. - V. 29, N 3. - P. 215-216.

19. Cannon L.O. Locating multiples of primes in Pascal's triangle // College Math. J. - 1989. - V. 20, - P. 324-328.

20. Carroll T., Hilton P., Pedersen J. Eulerian numbers, pseudo-Eulerian coefficients, and weighted sums in Pascal's triangle // Nieuw Arch. Wisk. - 1991. - V. 9, N 1. - P. 41-64.

21. Chu Wen-Chang. On the evaluation of some determinants with q-binomial coefficients // J. Syst. Sci. and Math. Sci. - 1988. - V. 8, N 4. - P. 361-366.

22. Das S.K., Deo N. Recontres graphs: a family of bipartite graphs // Fibonacci Quart. - 1987. - V. 25, N 3. - P. 250-262.

23. Davis K.S., Webb W.A. Pascal's triangle modulo 4 // Fibonacci Quart. - V. 29, N 1. - P. 79-83.

24. Dodd F., Peele R. Some counting problems involving the multinomial expansion // Math. Mag. - 1991. V. 64, N 2. P. 115-122.

25. Edwards A.W.F. Patterns and primes in Bernoulli's triangle // Math. Spectrum. - 1990-1991. - V. 23, N 4. - P. 105-109.

26. Fielder D.C., Alford C.O. On a conjecture by Hoggatt with extensions to Hoggatt sums and Hoggatt triangles // Fibonacci Quart. - 1989. - V. 27, N 2. - P. 160-168.

27. Fielder D.C., Alford C.O. An investigation of sequences derived from Hoggatt sums and Hoggatt triangles // Applications of Fibonacci Numbers. Vol. 3. - Dordrecht: Kluver Academic Publ., 1990. - P. 77-88.

28. Fielder D.C., Alford C.O. Pascal's triangle: top gun or just one of the gang // Applications of Fibonacci Numbers. Vol. 4. - Dordrecht: Kluver Academic Publ., 1991. - P.

29. Fjelstad P. Extending Pascal's triangle // Comput. Math. Appl. - 1991. - V. 21, N 9. - P. 1-4.

30. Flath D., Peele R. A carry theorem for rational binomial coefficients // Applications of Fibonacci Numbers. Vol. 4. - Dordrecht: Kluver Academic Publ., 1991. - P. 109-120.

31. Giambolo V., Mines R., Pangelley D.J. P-adic congruences between binomial coefficients // Fibonacci Quart. - 1991. - V. 29, N 2. - P. 114-119.

32. Goetgheluck P. Computing binomial coefficients // Amer. Math. Monthly. - 1987. - V. 94, N 4. - P. 360-365.

33. Goetgheluck P. On prime divisors of binomial coefficients // Math. Comp. - 1988. - V. 51. - P. 325-329.

34. Griggs J.R., Sun C.H.W. Determinants among the binomial and trinomial coefficients // Abstracts of papers presented to AMS. - 1991. - V. 12, N 1. - P. 20.

35. Hilton P., Pedersen J. Looking into Pascal's triangle: combinatorics, arithmetic, and geometry // Math. Mag. - 1987. - V. 60, N 5. - P. 305-316.

36. Hilton P., Pedersen J. Binomial coefficients in the Pascal hexagon // Kolloquinm Math. - Didaktik der Universität Bayreuth. - 1988. V. 14. - P. 3-24.

37. Hilton P., Pedersen J. Extending the binomial coefficients to preserve symmetry and pattern // Comput. Math. Appl. - 1989. - V. 17, N 1-3. - P. 89-102.

38. Hilton P., Pedersen J., Rosenthal W. Pascalian triangles and hexagons // Proc. Descartes Conference. San Jose St. Univ. (Nov., 1989). - P. 26-31.

39. Hodgson B.R. On some number sequences related to the parity of binomial coefficients // Fibonacci Quart. - 1992. - V. 30, N 1. - P. 35-47.

40. Holte J.M. A recurrence-relation approach to fractal dimension in Pascal's triangle // International Congress of Math. (Abstracts). - Japan, Kyoto. - 1990. - P. 227.

41. Holte J. The dimension of the set of multinomial coefficients not divisible by n // Abstracts of papers presented to AMS. - 1991. - V. 12, N 1. - P. 21.

42. Holte J.M. The fractal geometry of multinomial coefficients not divisible by n // NCS/MAA Workshop in Duluth, MN (Sep., 1991).

43. Howard F.T. Congruences for the Stirling numbers // Acta Arith. - 1990. - V. 55, N 1. - P. 29-41.

44. Howard F.T. Extensions of congruences of Glaisher and Nielsen concerning Stirling numbers // Fibonacci Quart. - 1990. - V. 28, N 4. - P. 355-362.

45. Howard F.T. Multinomial and q-binomial coefficients modulo 4 and modulo p // Fibonacci Quart. (to appear).

46. Imai Y., Seto Y., Tanaka S., Yutani H. An expansion of x^m and its coefficients // Fibonacci Quart. - 1988. - V. 26, N 1. - P. 33-39.

47. Ishikawa T. Super congruence for the Apery numbers // Nagoya Math. J. - 1990. - V. 118. - P. 195-202.

48. Knuth D.E., Wilf H.S. The power of a prime that divides a generalized binomial coefficient // J. Reine Angew. Math. - 1989. - T. 396. - S. 212-219.

49. Kubelka R. Decomposition of Pascal's kernels mod p^2 // Abstracts of papers presented to AMS. - 1990. - V. 11, N 1. - P. 15.

50. Kwong Y.H.H. Minimum periods of S(n,k) modulo M // Fibonacci Quart. - 1989. - V. 27, N 3. - P. 217-221.

51. Kwong Y.H.H. Minimum periods of binomial coefficients modulo M // Fibonacci Quart. - 1989. - V. 27, N 4. - P. 348-351.

52. Lakhtakia A., Messier R., Varadan V.V., Varadan V.K. Fractal dimension from the back-scattering cross section // J. Phys. A.: Math. Gen. - 1987. - V. 20, - P. 1615-1619.

53. Lakhtakia A. et al (see 52). Fractal structures derivable from the generalization of the Pascal triangle // J. Phys. A.: Math. Gen. - 1987. - V. 20. - P. L735-L738.

54. Lakhtakia A. et al (see 52). Fractal sequences derived from the self-similar extensions of the Sierpinski gasket // J. Phys. A.: Math. Gen. - 1988. - V. 21. - P.

55. Lakhtakia A. Khayyam's (Pascal's) triangle. Commentary on paper of C.A. Pickover (see 66). - Leonardo. - 1991. - V. 11. - P. 333-335.

56. Lee J.-Z., Lee J.-S. Some properties of binomial coefficients // Fibonacci Quart. - 1987. - V. 25, N 4. - P. 339-342.

57. Long C.T., Ando S. Two generalizations of Gould's star of David theorem // Fibonacci Quart. (to appear).

58. MacLeord R.A. Generalisation of a result of E. Lucas // Canad. Math. Bull. - 1988. - V. 31, N 1. - P. 95-98.

59. Nijenhuis A., Wilf H.S. Periodicities of partition functions and Stirling numbers modulo p // J. Number Theory. - 1987. - V. 25. - P. 308-312.

60. Passoja D.E., Lakhtakia A. Carpets and rugs: an exercise in numbers // Leonardo (to appear).

61. Passoja D.E., Lakhtakia A. Variations on a Persian theme // Irnt. Rec. Math. (to appear).

62. Peele R., Wilf H. Congruence problems involving Stirling numbers of the first kind // Abstracts of papers presented to AMS. - 1991. - V. 12, N 1. - P. 22.

63. Peitgen H.-O., Jürgens H. Fraktale: Gezähmtes Chaos. - München: Siemens, 1988. - 70 pp.

64. Philippou A.N., Georghiou C. Convolutions of Fibonacci-type polynomials of order k and negative binomial distributions of the same order // Fibonacci Quart. - 1989. - V. 27, N 3. - P. 209-216.

65. Philippou A.N., Makri F.S. Longest circular runs with an application in reliability via the Fibonacci-type polynomials of order k // Applications of Fibonacci Numbers. Vol. 3. - Dordrecht: Kluwer Academic Publ., 1990. - P. 281-286.

66. Pickover C.A. On the aesthetics of Sierpinski gaskets formed from large Pascal's triangles // Leonardo (to appear).

67. Reiter A. Patterns in Pascal's triangle. First prize entry. - Westinghouse Sci. Talent Search for 1990-91.

68. Robbins N. Representing $\binom{2n}{n}$ as a sum of squares // Fibonacci Quart. - 1987. - V. 25, N 1. - P. 29-33.

69. Rogers D.G., Shapiro L.W. Problem E3343 // Amer Math. Monthly. - 1989. - V. 96. - P. 734.

70. Singmaster D. Solution III of Problem 1255 // Math. Mag. - 1988. - V. 61. - P. 53-54.

71. Stein A.H. Binomial coefficients not divisible by a prime // Lect. Notes Math. - 1989. - N 1383. - P. 170-177.

72. Sun C.H.W. Binomial determinants with applications // Ph.D. Thesis: Univ. of South Carolina. - 1991. - 81 p.

73. Sved M. Divisibility - with visibility // Math. Intelligencer. - 1988. - V. 10, N 2. - P. 56-64.

74. Sved M., Pitman J. Divisibility of binomials by prime powers. A geometrical approach // ARS Combinatoria. A. - 1988. - V. 26. - P. - 197-222.

75. Sved M. Fractals, recursions, divisibility // Australasian J. Combinatorics. (to appear).

76. Sved M. The geometry of the binomial array modulo p^2 and p^3 // Discrete Math. (to appear).

77. Turner J.C. Convolution trees and Pascal-T triangles // Fibonacci Quart. - 1988. - V. 26, N 4. - P. 354-365.

78. Webb W.A. The number of binomial coefficients in residue classes modulo p and p^2 // Colloq. Math. - 1990. - V. 60-61, N 1. - P. 275-280.

79. Wong W.J. Powers of a prime dividing binomial coefficients // Amer. Math. Monthly. 1989. - V. 96, N 6. - P. 513-517.

80. Yeung K.M. On congruences for binomial coefficients // J. Number Theory. - 1989. - V. 33, N 1. - P. 1-17.

II

81. Abbott H.L., Erdös P., Hanson D. On the number of times an integer occurs as a binomial coefficient // Amer. Math. Monthly. - 1974. - V. 81, N 3. - P. 256-261.

82. Carlitz L. Some congruences involving binomial coefficients // Elem. Math. - 1959. - V. 14, N 1. - P. 11-13.

83. Carlitz L. Some determinants of q-binomial coefficients // J. Reine Angew. Math. - 1967. - V. 226. - S. 216-220.

84. Cartier P. Quelques remarques sur la divisibilité des coefficients binomiaux // Enseignement Math. - 1970. - V. 16, N 1. - P. 21-30.

85. Church C.A., Jr. On the enumeration of certain triangular arrays // Fibonacci Quart. - 1970. - V. 8, N 3. - P. 235-241.

86. Deligne P. Une congruence entre coefficients multi-nomiaux // Math. et Paed. - 1971. - T. 16, N 46. - P. 34-37.

87. Duncan D.R., Litwiller B.H. A pattern: Pascal's triangle and prime numbers // Math. Teacher. - 1975. - V. 68, N 1. - P. 23-26.

88. Erdös P., Graham R.L. Old and new Problems and Results in Combinatorial Number Theory. - Genève: L'Enseignement Math., 1980. - 128 p.

89. Fielder D.C. Generation of Stirling numbers by means of special partitions of numbers // Fibonacci Quart. - 1968. V. 6, N 5. - P. 1-10.

90. Gessel I. Some congruences for Apéry numbers // J. Number Theory. - 1982. - V. 14, N 3. - P. 362-368.

91. Gessel I. Some congruences for generalized Euler numbers // Canad. J. Math. - 1983. - V. 35, N 4. - P. 687-709.

92. Gould H.W., Greig W.E. A Lucas triangle primality criterion // Notices Amer. Math. Soc. - 1977. - V. 24, N 2. - P. A-231.

93. Harborth H. Dualzahlen im Pascal-Dreieck // Praxis Math. - 1971. - V. 13, N 1. - S. 76-77.

94. Hexel E. Einige Bemerkungen zum Pascal'schen Dreieck // Contributions to Graph Theory and its Applications. - Ilmenau: Tech. Hochschule Ilmenau, 1977. - P. 121-128.

95. Hexel E., Sachs H. Counting residues modulo a prime in Pascal's triangle // Indian J. Math. - 1978. - V. 20, N 2. - P. 91-105.

96. Hitotumatu S., Sato D. Simple proof that a p-adic Pascal's triangle is 120° rotatable // Proc. Amer. Math. Soc. - 1976. - V. 59, N 2. - P. 406-407.

97. Hoggatt V.E., Jr., Bicknell M. Fourth power Fibonacci identities from Pascal's triangle // Fibonacci Quart. - 1964. - V. 2, N 4. - P. 261-266.

98. Hoggatt V.E., Jr., Bicknell-Johnson M. Divisibility properties of polynomials in Pascal's triangle // Fibonacci Quart. - 1978. - V. 16, N 6. - P. 501-513.

99. Hudson R.H., Williams K.S. Cauchy-type congruences for binomial coefficients // Proc. Amer. Math. Soc. - 1982. - V. 85, N 2. - P. 169-174.

100. Kapur J.N. Generalized Pascal's triangles and algebraic identities // Math. Student. - 1982. - V. 43, N 3-4. - P. 361-377.

101. Labahn R. On an interesting property of binomial coefficients // Rostock Math. Kolloq. - 1986. - N 29. - P. 21-24.

102. Levitt H. An extension of Pascal's triangle // Math. Gaz. - 1961. - V. 45, N 352. - P. 99-107.

103. McClenon R.B. Pascal's arithmetical triangle // Proc. Iowa Acad. Sci. - 1956. - V. 63. - P. 534-537.

104. Monzingo M.G. On congruence modulo a power of a prime // Fibonacci Quart. - 1976. - V. 14, N 1. - P. 23-24.

105. Oehlmann D. Pascal'sche Dreiecke und Quersummen // Math. - Phys. Korresp. - 1982. - N 124. S. 13-18.

106. Parker R.V. Multiplication and division by binomial factors // Amer. Math. Monthly. - 1959. - V. 65, N 1. - P. 39-42.

107. Pignatoro S. Triangoli aritmetici e sviluppi di potenze // Rend. Accad. Sci. Fiz. Mat. Napoli. - 1973 (1974). - T. 40. - P. 21-25.

108. Price G.B. Distributions derived from the multinomial expansion // Amer. Math. Monthly. - 1946. - V. 53, N 2. - P. 59-74.

109. Radoux C. Quelques propriétés arithmétiques des nombres d'Apéry // C.R. Acad. Sci. Paris Sér. A-B. - 1980. - V. 291, N 10. - P. A567-A569.

110. Roberts J.B. On binomial coefficient residues // Canad. J. Math. - 1957. - V. 9, N 3. - P. 363-370.

111. Rogers D.G. A Schröder triangle: three combinatorial problems // Lecture Notes in Mathematics. - Berlin: Springer-Verlag, 1977. - Vol. 622. - P. 175-196.

112. Rogers D.G. Triangular arrays associated with some partitions // A Collection of Manuscripts Related to the Fibonacci Sequence - 18th Anniversary Volume. - Santa Clara: FA, 1980. - P. 169-173.

113. Rogers D.G. Epletts's identity for renewal arrays // Discrete Math. - 1981. - V. 36. - P. 97-102.

114. Rose N.J. The Pascal triangle and Sierpinski's tree. - Math. Calendar 1981. - Raleigh, NC: Rome Press, 1980.

115. Ruderman H.D. Problem 1255 // Math. Mag. - 1986. - V. 59, n 4. - P. 297.

116. Rusu E. Conditia ca C_n^k să fie multiplu de numarul prim p // Gaz. Mat. Ser. A. - 1954. - T. 6, F. 1. - S. 30-33.

117. Salmeri A. Introduzione alla teoria dei coefficienti fattoriali // Giorn. Mat. Battaglini. - 1962. - T. 90, F. 1. - S. 44-54.

118. Seymour Dale. Visual Patterns in Pascal's Triangle. - Dale Seymour Publications, 1986.

119. Shannon A.G. A recurrence relation for generalized multinomial coefficients // Fibonacci Quart. - 1979. - V. 17, N 4. - P. 344-347.

120. Singmaster D. Divisibility of binomial and multinomial coefficients by primes and prime powers // A Collection of Manuscripts Related to the Fibonacci Sequence - 18th Anniversary Volume. - Santa Clara: FA, 1980. - P. 98-113.

121. Sved M. Geometry of combinatorial arithmetic // Ars Combinatoria. - 1986. - V. 21-A. - P. 271-298.

122. Tonne P.C. A regular determinant of binomial coefficients // Proc. Amer. Math. Soc. - 1973. - V. 41, N 1. - P. 17-23.

123. Torrento R.F., Fuchs J.A. Generalized binomial coefficients // Fibonacci Quart. - 1964. - V. 2, N 4. - P. 296-302.

124. Toschi, L.M. The permutations triangle // Math. Teacher. - 1974. - V. 67, N 3. - P. 210-214.

125. Waters W.M., Jr. An alternate formula for the Jth coefficient of $(a+b)^{-n}$ and a surprising relationship to the triangle of Pascal // MATYC J. - V. 15, N 2. - P. 122-125.

126. Weisman C.S. Some congruences for binomial coefficients // Michigan Math. J. - 1977. - V. 24, N 2. - P. 141-151.