Likewise, denote by B_r the isosceles trapezoid whose height, widths of upper and lower bases, and row numbers of the upper and lower bases are, respectively, $h_r=\frac{1}{2}(3^r-1)$, $d_U=2$, $d_L=3^r-1$, $n_U=\frac{1}{2}(3^r+1)$, $n_L=3^r-1$, $r\geq 1$. In Figure 28 one can distinguish the triangles $A_0,\ldots,A_3$, and the trapezoids B_1,B_2.

Theorem 3.9. Let the row number of the base of the generalized Pascal triangle of order 3 be $n=(3^r-1)/2$. Then for any natural number r, the number of trinomial coefficients not divisible by three is given by

$$Q_{1,2}(n) = 2^{r-1}(3^r+1). \qquad (3.12)$$

Proof: Consider the triangle A_r. The row $n=(3^{r-1}-1)/2$ is inside A_r and is the base of the triangle A_{r-1}. The following row number, being one greater, has a ternary representation $(1\ 1\ldots1\ 2)_3$ which contains one $<2>_1$ block. From Theorem 3.3, it follows that in this row only four trinomial coefficients are not divisible by three: $\binom{n}{m}_3$ for $m=0,1$ and $m=3^{r-1}$, $3^{r-1}+1$; these generate two trapezoids B_{r-1} with lower bases on the row $n=3^{r-1}-1$. Now it follows from Theorem 3.3 that in row $n=3^{r-1}$ there are only three nonzero coefficients (ones): $\binom{n}{m}_3$ for $m=0,\ 3^{r-1},\ 2\cdot 3^{r-1}$; these generate three triangles A_{r-1}, whose bases coincide with the base of A_r. It follows that A_r itself represents a geometric sum of four triangles A_{r-1} and two trapezoids B_{r-1} (Figure 29a). In like fashion, it may be shown that B_r is a geometric sum of four trapezoids B_{r-1} and two triangles A_{r-1} (Figure 29b). Denote by a_r the number of trinomial coefficients in A_r not divisible by three, and by b_r the same for B_r. By the preceding arguments we may form the system

$$\left.\begin{aligned} a_r &= 4a_{r-1} + 2b_{r-1} \\ b_r &= 2a_{r-1} + 4b_{r-1} \end{aligned}\right\}, \tag{3.13}$$

where $r \geq 2$ and the initial data is $a_1 = 4$, $b_1 = 2$. The solution of (3.13) is

$$a_r = 2^{r-1}(3^r+1), \quad b_r = 2^{r-1}(3^r-1), \tag{3.14}$$

the first of which gives $Q_{1,2} = a_r$, and the theorem is proved.

The total number of trinomial coefficients in A_r is $(3^r+1)^2/4$, and so the number of coefficients divisible by three is given by

$$Q_3((3^r-1)/2) = \tfrac{1}{4}(3^r+1)(3^r-2^{r+1}+1). \tag{3.15}$$

It is not difficult to see that from some n onward $Q_3(n) >> Q_{1,2}(n)$.

Theorem 3.10. For $n \to \infty$, $\lim Q_{1,2}(n)/Q_3(n) = 0$.

Proof: Since $Q_{1,2}$ and Q_3 are nondecreasing functions of n, for $(3^r-1)/2 \leq n < (3^{r+1}-1)/2$ we have

$$[Q_{1,2}(n)\,/\,Q_3(n)] < \left[Q_{1,2}\left(\frac{3^{r+1}-1}{2}\right) \Big/ Q_3\left(\frac{3^r-1}{2}\right)\right].$$

Thus,

$$\lim_{n\to\infty} [Q_{1,2}(n)\,/\,Q_3(n)] < \lim_{r\to\infty} \left[Q_{1,2}\left(\frac{3^{r+1}-1}{2}\right) \Big/ Q_3\left(\frac{3^r-1}{2}\right)\right].$$

Using (3.12) and (3.15), we find

$$[Q_{1,2}(n)\,/\,Q_3(n)] < 12 \Big/ \left[\left(\frac{3}{2}\right)^r + \left(\frac{1}{2}\right)^r - 2\right].$$

But as $r \to \infty$, $\left(\frac{3}{2}\right)^r \to \infty$, $\left(\frac{1}{2}\right)^r \to 0$, and we have

$$\lim_{n \to \infty} [Q_{1,2}(n) / Q_3(n)] = 0,$$

which proves the theorem. The method of blocks introduced here may also be applied to finding distribution of the coefficients $\binom{n}{m}_s$ in the triangle of order s, for p=2,3,...; the calculations, of course, become increasingly complicated.

In Figures 30a and 30b we show the distributions of the trinomial coefficients with respect to the moduli 2^v and 3^v, respectively, i.e., the highest power which divides the given coefficients.

```
                              0
                            0 0 0
                          0 1 0 1 0
                        0 0 1 0 1 0 0
                      0 2 1 4 0 4 1 2 0
                    0 0 0 1 0 0 0 1 0 0 0
                  0 1 0 1 1 1 0 1 1 1 0 1 0
                0 0 2 0 0 1 0 0 0 1 0 0 2 0 0
              0 3 2 4 1 3 4 3 0 3 4 3 1 4 2 3 0
            0 0 0 2 1 1 1 8 0 0 0 8 1 1 1 2 0 0 0
          0 1 0 1 0 2 1 2 0 1 0 1 0 2 1 2 0 1 0 1 0
        0 0 1 0 3 0 0 1 0 0 1 0 2 0 0 1 0 0 3 0 1 0 0
      0 2 1 5 0 3 1 2 1 2 1 3 0 3 1 2 1 2 1 3 0 5 1 2 0
    0 0 0 1 0 0 0 1 5 1 2 1 0 0 0 1 2 1 5 1 0 0 0 1 0 0 0
  0 1 0 1 3 1 0 1 0 3 1 4 0 1 0 1 0 4 1 3 0 1 0 1 3 1 0 1 0
0 0 3 0 0 2 0 0 1 0 0 1 0 0 1 0 1 0 0 1 0 0 1 0 0 2 0 0 3 0 0
```

Figure 30a

0

0 0 0

0 0 1 0 0

0 1 1 0 1 1 0

0 0 0 0 0 0 0 0 0

0 0 1 1 2 1 2 1 1 0 0

0 1 1 0 2 2 1 2 2 0 1 1 0

0 0 0 0 0 0 1 1 1 0 0 0 0 0 0

0 0 2 0 0 2 0 0 2 0 0 2 0 0 2 0 0

0 2 2 1 2 2 1 2 2 0 2 2 1 2 2 1 2 2 0

0 0 0 1 1 1 1 1 1 0 0 0 1 1 1 1 1 1 0 0 0

0 0 1 0 0 1 2 1 2 0 0 1 0 0 2 1 2 1 0 0 1 0 0

0 1 1 0 1 1 0 2 1 0 1 1 0 1 1 0 1 2 0 1 1 0 1 1 0

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

0 0 1 1 1 1 1 1 1 1 3 1 1 1 1 1 1 1 3 1 1 1 1 1 1 1 1 0 0

0 1 1 0 4 3 2 1 5 1 2 2 1 4 3 2 3 4 1 2 2 1 5 1 2 3 4 0 1 1 0

Figure 30b

3.2 DIVISIBILITY AND DISTRIBUTION MODULO p OF FIBONACCI, LUCAS, AND OTHER NUMBERS IN ARITHMETIC TRIANGLES

As we know, the Fibonacci numbers are defined by the recurrence relation

$$F_{n+1} = F_n + F_{n-1}, \quad F_1 = 1, \quad F_2 = 1. \tag{3.16}$$

Interpretations of these numbers may be found, for example, in [20], and they play an early part in number theory and combinatorial analysis. They form the sequence 1,1,2,3,5,8,13,..., and may be studied from the point of view of sequences, or may be constructed from their arithmetic triangle as in section 1.4.

Studies on questions of divisibility and the distribution modulo the prime p of the Fibonacci numbers may be found in [11, 62, 171, 199, 245, 286, 339, 387-389, 390]. We note below some of these results concerning the above topics, as well as periodicity modulo p.

The paper of V.E. Hoggatt and G.E. Bergum [199] gives several results:

(1) if $p \geq 2$ is a prime and F_n is divisible by p, then for $s \geq 1$, $F_{np^{s-1}}$ is divisible by p^s;

(2) if $n=3^m \cdot 2^{s+1}$, where $m,s \geq 1$, then F_n is divisible by n;

(3) if $n=3^m \cdot 2^{s+1} \cdot 5^r$, where $m,s,r \geq 1$, then F_n is divisible by n.

The sequence of numbers $\{x_n\}$, $n \geq 1$, is said to be uniformly distributed mod m, where $m \geq 2$ is a whole number, if

$$\lim_{n \to \infty} \frac{1}{n} A(n,m,r) = \frac{1}{m}$$

for each $r=0,1,\ldots,m-1$, where A(n,m,r) is the number of terms of the sequence congruent to r (mod m). L. Kuipers and J.S. Shine [245] showed that:

(1) the Fibonacci sequence $\{F_n\}$ is uniformly distributed mod 5;

(2) the sequence $\{F_n\}$ is non-uniformly distributed mod p, where $p \geq 2$ is any prime except $p=5$;

(3) the sequence $\{F_n\}$ is non-uniformly distributed with respect to any composite modulus m, if $m \neq 5^k$, $k=3,4,\ldots$.

The sequence $\{L_n\}$, defined by the recurrence relation (3.16) but with initial conditions $L_1=1$, $L_2=3$, is discussed in [199], where it is shown that for $n=2 \cdot 3^k$, $k \geq 1$, L_n is divisible by n.

In [129], it is proved that the Catalan number $\binom{2n}{n}/(n+1)$ is odd if $n=2^r-1$, where r is a nonnegative whole number. And in [60], it is shown that for any prime $p>2$, the Catalan sequence $\{C_n\}$ may be decomposed into blocks of successive values C_n divisible by p (block B_k), and not divisible by p (block $\bar{B}_k$), with respective lengths l_k and $\bar{l}_k$; further

$$l_k = \frac{1}{2}(p^{m+1-\delta}-3), \quad \bar{l}_k = \frac{1}{2}(p+3+6\delta),$$

where m is the largest natural number for which k is divisible by $\left(\frac{p+1}{2}\right)^m$, $\delta=\delta_{3p}$, and

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}.$$

The present author in [11] studied the distribution modulo p of the elements in the Fibonacci and Lucas triangles. In Figure 31 are shown the distributions of the elements in the Fibonacci triangle for $p=2$ (31a) and $p=3$ (31b), and in Figure 32 these same distributions for the Lucas triangle.

Divisibility questions for the sequence of generalized Fibonacci numbers are discussed in [182].

We show also the distributions modulo $p=2$ or 3, as indicated in the figures, of the elements in the arithmetic triangles [366-369] composed of Gaussian binomial coefficients (Figure 33), Euler numbers (Figure 34), Stirling numbers of the first (Figure 35) and second (Figure 36) kinds, and (Figure 37) the distribution in the Pascal triangle of elements divisible by their row numbers, indicated by O's (divisible) and X's (not divisible) [177].

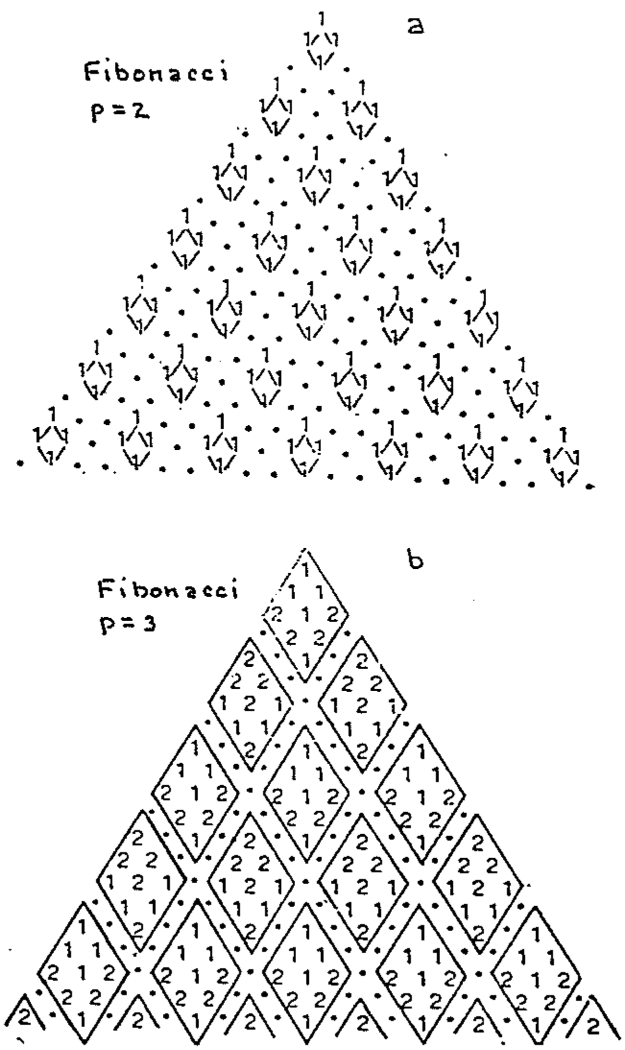

Figure 31

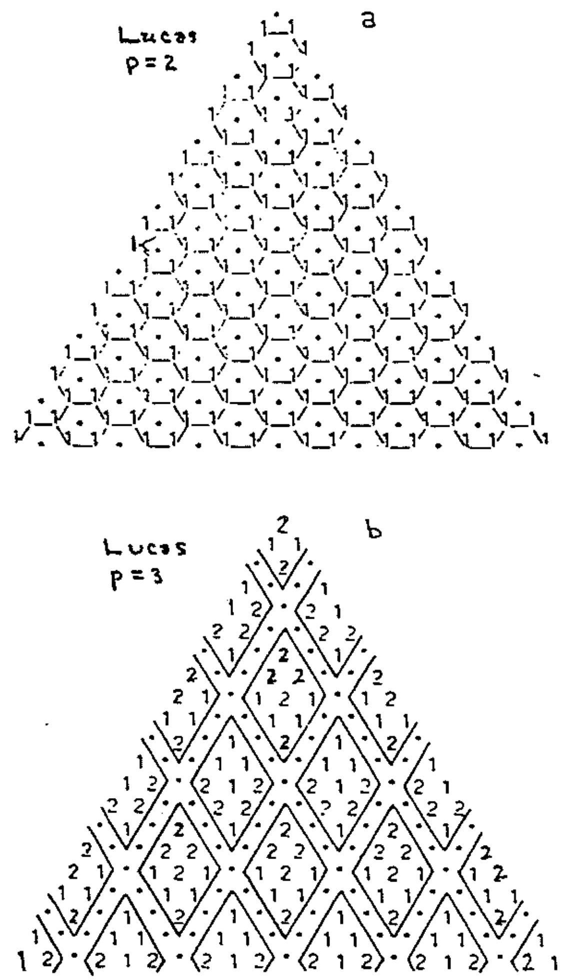

Figure 32

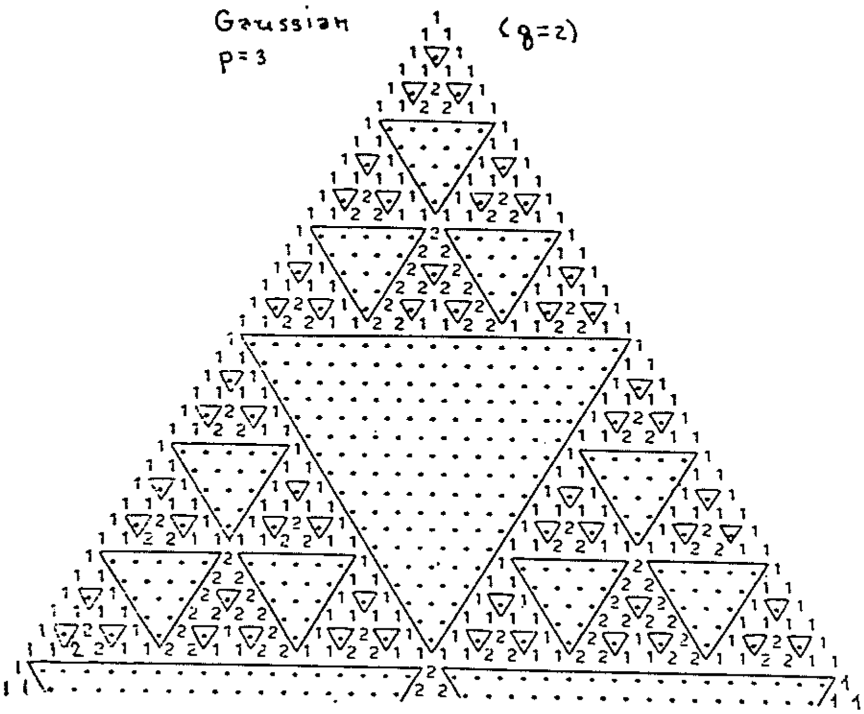

Figure 33

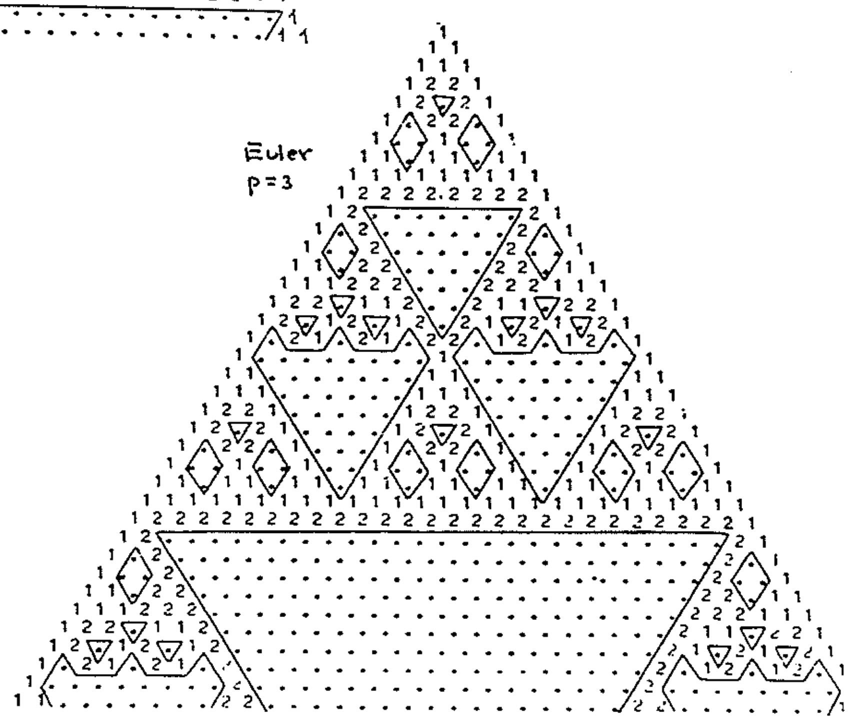

Figure 34

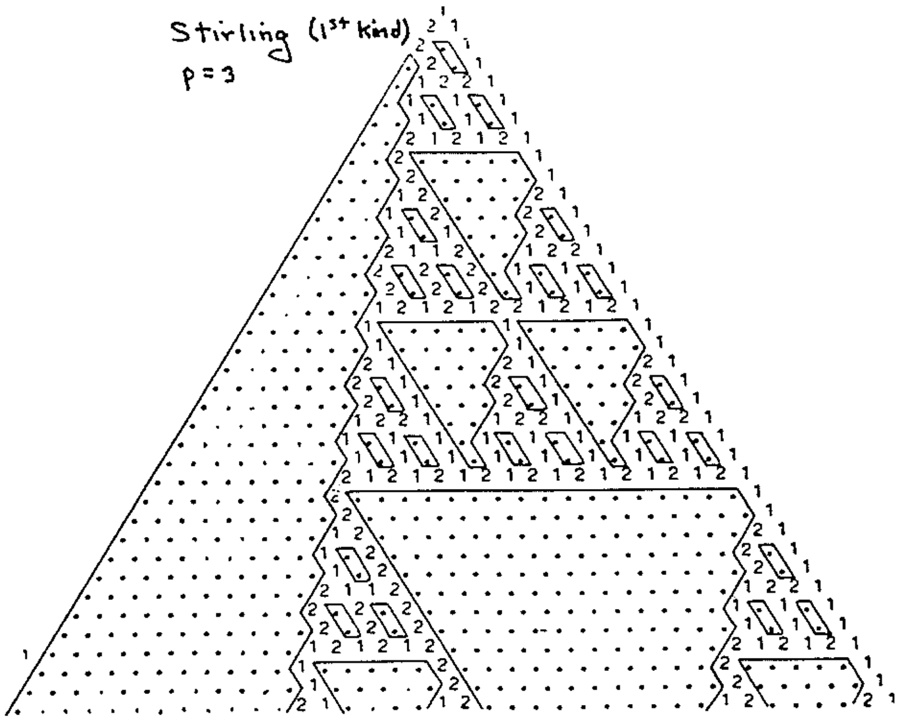

Figure 35

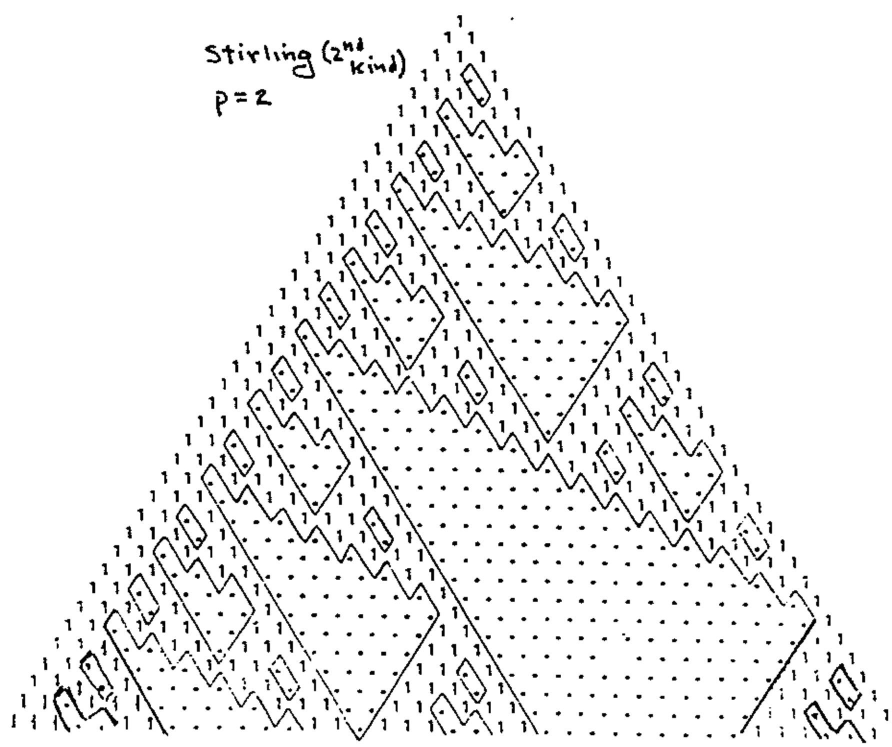

Figure 36

```
                 X
                O O
               X O X
              X O O X
             X O X O X
            X O O O O X
           X O X X X O X
          X O O O O O O X
         X O X O X O X O X
        X O O X O O X O O X
       X O X O O X O O X O X
      X O O O O O O O O O O X
     X O X X X O O O X X X O X
    X O O O O O O O O O O O O X
   X O X O X O X X X O X O X O X
  X O O X O X X O O X X O X O O X
 X O X O X O X O X O X O X O X O X
X O O O O O O O O O O O O O O O O X
```

Figure 37

3.3 DISTRIBUTION OF SOME NUMERICAL SEQUENCES MODULO p

Material on questions of divisibility and distribution mod p of various numerical sequences appears in [68, 106, 107, 266, 299, 306-307, 332, 340, 377, 378, 386, 402, 404]. Some of these discuss sequences which are related one way or another to binomial coefficients, Fibonacci numbers, Lucas numbers, and other special numbers.

As is known, if p is prime, the natural number n is divisible by $(p-1)$ if

$$r^n \equiv 1 \pmod{p}, \quad r = 1,2,\ldots,p-1.$$

M. Bhaskaran [68] showed, for $p>2$, that the positive whole number n is divisible by $(p+1)$ if

$$\sum_{k=0}^{\left[\frac{n-r}{p-1}\right]} (-1)^k \binom{n}{r+kp-k} \equiv 0 \pmod{p}, \quad r = 1,3,\ldots,p-2.$$

A. Rotkiewicz proved in [332] that for any natural number $a>1$ there exists an n such that a^n+1 is divisible by n.

[106, 107, 404] are devoted to questions of divisibility of the sequence given by the recurrence

$$A_{n+2} = aA_{n+1} + bA_n,$$

for various choices of a,b and initial conditions A_0, A_1. Thus, in [106], $a=P$, $b=-Q$, P and Q whole numbers, and $A_0=0$, $A_1=1$; in [107], $a=p+2$, $b=-(p+1)$, p a prime, $A_0=0$, $A_1=1$, and it is shown that the sequence is uniformly distributed mod p^s.

In [378] there is discussed a sequence, and its properties, formed from the fifth column of the Pascal triangle.

In [386] the sequence of triangular numbers is discussed with respect to the modulus n and it is shown that the sequence is periodic mod n (n odd) and mod 2n (n even). The periodicity of m-gonal numbers mod n is also treated in [340].

A. Perelli and U. Zannier [306-307] studied arithmetic properties and periodicity mod p of some sequences; they proved, for example, that if

$$f(n+p) \equiv f(n) \pmod{p}, \quad p>p_0,\ n\in N,$$

then for certain specified conditions f must be a polynomial.