CHAPTER 5

GENERALIZED ARITHMETIC GRAPHS AND THEIR PROPERTIES

In this chapter we consider the properties of generalized arithmetic graphs, special cases of which are graphs modeled on generalized Pascal triangles. We will look at planar and spatial graphs of generalized Pascal triangles of order m, prove a theorem on their isomorphisms, and give an asymptotic formula for the number of paths in these graphs as $k \to \infty$. We also prove a theorem on the cross sections of spatial arithmetic graphs.

5.1 GENERALIZED ARITHMETIC GRAPHS

A technique often used in the solution of combinatorial problems is to interpret or reformulate the problem in the form of a graph, which provides a visual interpretation of the combinatorial object and may make possible the discovery of new properties. We might cite, for example, the use of the Ferrers graph for the representation of partitions [41], and the use of graphs in the study of partially ordered sets [39].

The interpretation in the form of a graph, of objects essentially combinatorial, allowed H. Hosoya [219, 220] to discover connections between the elements of the Pascal triangle and the Fibonacci sequence, and a number of forms of chemical structures. In [19] the possibility of using the Pascal triangle in building models of genetic codes is considered. The graph interpretation is also used in [29], in which a study of the properties of the graph suggests new ways and algorithms for the solution of combinatorial problems.

S.K. Das, N. Deo, and M.J. Quinn [113-116], and B.P. Sinha et al. [353], introduced and studied the properties of the Pascal graph; the adjacency matrix of the vertices of the graph coincides with the Pascal triangle mod 2. In particular, they introduced the Pascal matrix, which is a symmetric matrix with zeros on the main diagonal, and below (and above) it the Pascal triangle mod 2. They also studied and determined the properties of the Pascal planar graph.

We will introduce the idea of the generalized arithmetic graph and prove a theorem on the number of its paths; special cases of these graphs are the graphs modeled on various modifications of the Pascal triangle.

A generalized arithmetic graph G(F,X) (Figure 55) is a regular Berge graph [26] with a set X of vertices and mapping F which associates with each vertex $x \in X$ a subset (possibly empty) of X, i.e.,

$$F_x = \left\{ y \mid y \in X \wedge \vec{J}(x,y) \right\},$$

where $\vec{J}(x,y)$ is the edge running from the vertex $x \in X$ to the vertex $y \in X$.

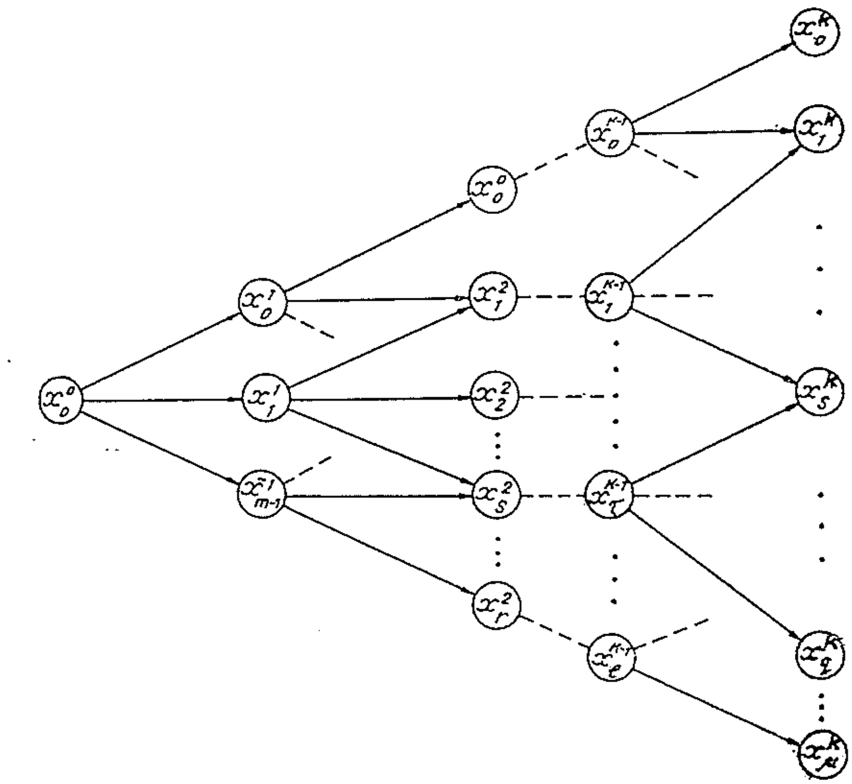

Figure 55

Definition 5.1. The notation

$$B_0=\{\alpha_0^0\},\ B_1=\{\alpha_0^1,\ \alpha_1^1,\ \ldots,\ \alpha_{m_1-1}^1\},\ B_2=\{\alpha_0^2,\ \alpha_1^2,\ \ldots,\ \alpha_{m_2-1}^2\},\ \ldots,\ B_k=\{\alpha_0^k,\ \alpha_1^k,\ \ldots,\ \alpha_{m_k-1}^k\}$$

will denote a basis (set) for the generalized arithmetic graph, with the cardinality of the indexed sets given by

$$|B_0| = 1, \quad |B_1| = m_1, \quad |B_2| = m_2, \ldots, \quad |B_k| = m_k .$$

Definition 5.2. A generalized arithmetic graph is an oriented graph, for which the following conditions hold:

(a) $X = \bigcup_{i=0}^{k} X_i$, and $X_i \cap X_j = \varnothing$, i.e., subsets of vertices at different levels have no common vertices;

(b) $\exists!\, x_0 \in X\left[Fx_0 = X_1 \wedge X_1 = B_1 \wedge F^{-1}x_0 = \varnothing\right]$, i.e., there exists a unique vertex $x_0 \in X$ for which the assertion is true, and this vertex is the root of the graph G(F,X);

(c) $\forall x_s \in X_{i-1}\left[Fx_s \subseteq X_i \rightarrow Fx_s = \left\{x_s + \alpha_0^i,\ x_s + \alpha_1^i,\ \ldots,\ x_s + \alpha_{m_i-1}^i\right\}\right]$, i.e., from each vertex $x_s \in X_{i-1}$ there issue exactly $|B_i|$ edges;

(d) $\forall x_s \in X_k[Fx_s = \varnothing]$, i.e., any vertex $x_s \in X_k$ at level k is terminal, and the subset X_k is terminal.

Lemma 5.1. In the graph G(F,X), if the vertex $x_s \in X_j$ is reachable from $x_j \in X_i$, then these vertices are joined by a path of length $(j-i)$.

Proof. Since the graph is a Berge graph it contains no loops or inconsistently oriented edges, i.e.,

$$\forall x_q \in X_i\left[Fx_q \in X_{i+1} \wedge x_q \,\overline{\in}\, Fx_q\right].$$

If $x \in X_i$, then the edge $\vec{J}(x,y)$ connects the vertex $x \in X_i$ with the vertex $y \in X_{i+1}$, and this edge is uniquely determined (Definition 5.2(c)), and, we may assert that the graph G(F,X) is of increasing character. Thus if the vertex $x_s \in X_j$ is reachable from the vertex $x_q \in X_i$, then the vertices are connected by a path of length $(j-i)$, which is what was to be shown.

It is known that any graph is completely determined, up to isomorphisms, by the adjacency matrix of its vertices. We will discuss below a number of properties of the graph G(F,X) in terms of the properties of its adjacency matrix.

Theorem 5.1. The number of paths connecting the vertex $x_0 \in X_0$ with the reachable vertex $x_s \in X_k$ of the generalized arithmetic graph is determined by the relations

$$r^0_{x_0,x_0} = 1,$$

$$r^1_{x_0,x_s} = \begin{cases} 1, \text{ if } x_s \in X_1 \\ 0, \text{ if } x_s \overline{\in} X_1 , \end{cases}$$

$$r^k_{x_0,x_s} = r^{k-1}_{x_0,x_s-\alpha^{k-1}_0} + r^{k-1}_{x_0,x_s-\alpha^{k-1}_1} + \cdots + r^{k-1}_{x_0,x_s-\alpha^{k-1}_{m_k-1}} ,$$

where $r^{\ell}_{i,j}$ is the (i,j) element of R^{ℓ}, the ℓ^{th} power of the adjacency matrix, $\ell = 0,1,2,\ldots,k$; $i = x_0$; $j = x_s \in X_k$.

Proof. By Lemma 5.1, the paths joining $x_0 \in X$ to $x_s \in X_k$ have a unique length k, and their number may be calculated with the help of the adjacency matrix R of the graph G(F,X). To do this we need to calculate the k^{th} power of the matrix $R = [r_{ij}]$, where we impose on the semi-ring K generated, the conditions [26] $\xi\eta = 1$, $\eta\xi = \theta^2 = \zeta = 0$. The distributivity condition must be fulfilled for elements of the form $n = 1 + 1 + \cdots + 1$, where the number is composed of units, and so k contains as a sub-semi-ring k′ the nonnegative whole numbers with ordinary addition and multiplication.

The construction of the matrix R of the graph G(F,X) takes the following form: we consider the subsets in the order $X_0, X_1 \ldots, X_k$, and the elements within a subset in order of increasing element number, and then index the rows and columns in the corresponding order. From Definition 5.2(b) it follows that

$$r^1_{x_0,\alpha^1_0} = r^1_{x_0,\alpha^1_1} = \cdots = r^1_{x_0,\alpha^1_{m_1-1}} = 1 , \tag{5.1}$$

i.e., the vertex x_0 (the root) is connected to the vertices of $X_1=B_1$ by paths of length one.

We adopt the convention that x_0 is joined to itself by a path of length one, i.e., $r^1_{x_0,x_0}=1$.

Then we can write (5.1) in the form

$$r^1_{x_0,x_s} = \begin{cases} 1, \text{ if } x_s \in X_1, \\ 0, \text{ if } x_s \bar{\in} X_1. \end{cases}$$

The elements of the matrix $R^2=r^2_{i,j}$ are defined as the number of paths of length two,

connecting $x_0 \in X_0$ with $j \in X_2$. To determine the elements of R^2 we use the formula

$$r^2_{i,j} = \sum_{q \in X} r^1_{i,q}\, r^1_{q,i}. \tag{5.2}$$

Since the vertex $j \in X_2$, by Definition 5.2(c), is reachable only from the vertices

$j-\alpha^2_0$, $j-\alpha^2_1$, ..., $j-\alpha^2_{m_2-1}$, and not from the remaining vertices of X_1,

$$r^1_{i,j-\alpha^1_i} \cdot r^1_{j-\alpha^1_i,j} = \begin{cases} r^1_{j-\alpha^1_{i,j}} = 1 \text{ and } r^1_{i,j-\alpha^1_i} \neq 0, \text{ if } (j-\alpha^1_i) \in X_1; \\ r^1_{j-\alpha^1_{i,j}} = 0 \text{ and } r^1_{i,j-\alpha^1_i} = 0, \text{ if } (j-\alpha^1_i) \bar{\in} X_1. \end{cases}$$

Thus, (5.2) may be written as

$$r^2_{i,j} = r^1_{i,j-\alpha^1_0} + r^1_{i,j-\alpha^1_1} + \cdots + r^1_{i,j-\alpha^1_{m_1-1}}. \tag{5.3}$$

Setting $i=x_0$ and $j=x_s$ in (5.3), and using Theorem 5.1 for the case k=2, we get

$$r^2_{x_0,x_s} = r^1_{x_0,x_s-\alpha^1_0} + r^1_{x_0,x_s-\alpha^1_1} + \cdots + r^1_{x_0,x_s-\alpha^1_{m_1-1}}. \tag{5.4}$$

Now, we assume that the conditions of the theorem are true for the (k-1)st power of the matrix R, and use induction to show their correctness for the kth power, $R^k=[r^k_{i,j}]$. The elements $r^k_{i,j}$ are determined by the formula

$$r^k_{i,j} = \sum_{q \in X} r^{k-1}_{i,q}\, r^1_{q,j} . \tag{5.5}$$

If we take into account that the vertex $j \in X_k$ is reachable (by paths of length one) only from the vertices

$$\left(j-\alpha_0^{k-1}\right),\ \left(j-\alpha_1^{k-1}\right),\ \ldots,\ \left(j-\alpha_{m_{k-1}-1}^{k-1}\right) \in X_{k-1},$$

we must have that the corresponding elements $r^1_{q,j}$ equal one for the subscript j, and equal zero for other vertices. Then, (5.5) may be written as

$$r^k_{i,j} = r^{k-1}_{i,j-\alpha_0^{k-1}} + r^{k-1}_{i,j-\alpha_1^{k-1}} + \cdots + r^{k-1}_{\ell,j-\alpha_{m_k-1}^{k-1}} ,$$

and from this it follows, setting $i=x_0 \in X_0$ and $j=x_s \in X$, that

$$r^k_{x_0,x_s} = r^{k-1}_{x_0,x_s-\alpha_0^{k-1}} + r^{k-1}_{x_0,x_s-\alpha_1^{k-1}} + \cdots + r^{k-1}_{x_0,x_s-\alpha_{m_k-1}^{k-1}} , \tag{5.6}$$

which is the assertion of the theorem.

Of particular interest in the solution of many combinatorial problems and complicated system design problems is the special case of the generalized arithmetic graph - the generalized m-arithmetic graph, with the basis $<\alpha_0,\alpha_1,\ldots,\alpha_{m-1}>$. It follows from Definition 5.2 that the generalized m-arithmetic graph may be obtained from the generalized arithmetic graph G(F,X) if the following conditions are satisfied:

- the generalized m-arithmetic graph has a unique basis $B=\{\alpha_0,\alpha_1,\ldots,\alpha_{m-1}\}$, where

$$B_1 = B_2 = \cdots = B_k = B \text{ and } |B| = m,$$

- the elements of the subsets $X_0, X_1, \ldots, X_k \subseteq X$, of vertices of the m-arithmetic graph are related among themselves by

$$\forall x_s \in X_{i-1}\left[Fx_s \subseteq X_i \rightarrow Fx_s = \{x_s+\alpha_0,\ x_s+\alpha_1,\ \ldots,\ x_s+\alpha_{m-1}\}\right].$$

The number of paths joining the (root) vertex $x_0 \in X_0$ with the reachable vertex $x_s \in X_k$ of the generalized m-arithmetic graph with the basis $<\alpha_0,\ldots,\alpha_{m-1}>$ satisfies the recurrence relations

$$r^0_{x_0,x_0} = 1, \tag{5.7}$$

$$r^1_{x_0,x_0} = \begin{cases} 1, \text{ if } x_s \in X_1, \\ 0, \text{ if } x_s \overline{\in} X_1, \end{cases} \tag{5.8}$$

$$r^k_{x_0,x_s} = r^{k-1}_{x_0,x_s-\alpha_0} + r^{k-1}_{x_0,x_s-\alpha_1} + \cdots + r^{k-1}_{x_0,x_s-\alpha_{m-1}}, \tag{5.9}$$

where (5.7) - (5.9) are obtained from the corresponding relation in Theorem 5.1, if we consider that the graph in this case has a unique basis $B=\{\alpha_0,\ldots,\alpha_{m-1}\}$.

5.2 THE SPECIAL CASE OF THE GENERALIZED m-ARITHMETIC GRAPH

The graph interpretation of the generalized Pascal triangle of order m is discussed in [3, 33-37]. Graphs of this type have been successfully used in decision problems arising in multi-stage discrete processes. Using the properties of the generalized arithmetic graph to construct interactive systems for technological decision making, it is possible to treat decision problems involving choices of equipment to achieve flexible industrial systems. References [33-35] give some algorithms for recognizing paths in the generalized arithmetic graph.

We consider now the special case of the generalized m-arithmetic graph with the basis $<\alpha_0,\alpha_1,\ldots,\alpha_{m-1}>$, which is a widely used model in recent studies of analogs of the Pascal triangle.

Let the elements of the basis $<\alpha_0,\ldots,\alpha_{m-1}>$ of the generalized arithmetic graph be defined in the following way:

$$\alpha_m = \alpha_{m-1} + p = \alpha_0 + mp = m,$$

where $\alpha_0=0$, $\alpha_1=1$.

<u>Definition 5.3</u>. The generalized m-arithmetic graph with the basis $<0,1,\ldots,m-1>$ is said to be a graph model of the generalized Pascal triangle of order m.

The recurrence formulas determining the number of paths in the m-arithmetic graph (Theorem 5.1), taking into account the properties of the basis, coincide with the recurrence formulas for the elements of the triangle of order m:

$$r^0_{x_0,x_0} = 1,$$

$$r^1_{x_0,x_s} = \begin{cases} 1, \text{ if } x_s \in X_1, \\ 0, \text{ if } x_s \overline{\in} X_1, \end{cases}$$

$$r^k_{x_0,x_s} = r^{k-1}_{x_0,x_s} + r^{k-1}_{x_0,x_{s-1}} + \cdots + r^{k-1}_{x_0,x_{s-m+1}}.$$

Example. Let m=3 and k=2, so that $X_0=\{0\}$, $X_1=\{0,1,2\}$, $X_2=\{0,1,2,3,4\}$. In this case the elements of the adjacency matrix, $r^2_{x_0,x_s}$, coincide with the coefficients generated by the function $A(t)=(1+t+t^2)^2$, i.e.,

$$A(t)=\sum_{x_s\in X_2} r^2_{x_0,x_s}\, t^{x_s}.$$

The geometric interpretation of the m-arithmetic graph corresponding to the generalized Pascal triangle of order 3 is shown in Figure 56a.

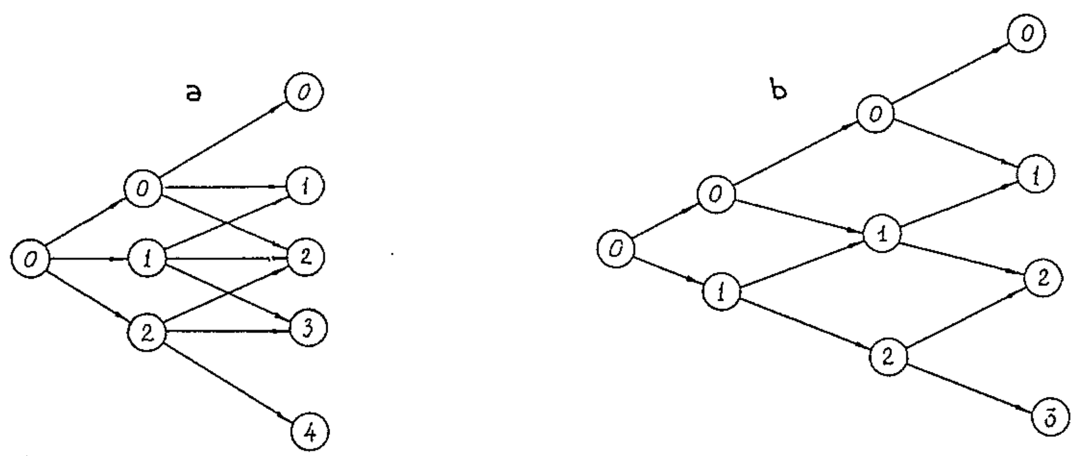

Figure 56

Definition 5.4. The generalized m-arithmetic graph with the basis <0,1> is said to be the graph of the Pascal triangle.

Theorem 5.2. The number of paths connecting the (root) vertex $x_0\in X_0$ to the vertex $x_s\in X_k$ in the graph of the Pascal triangle is given by the binomial coefficient $\binom{k}{x_s}$.

Proof. From Theorem 5.1, we have $r^0_{x_0,x_0}=\binom{0}{x_0}=1$. For k=1 the vertices in the subset $X_1=\{0,1\}$ are connected with x_0 by paths of length one; the numbers of paths are given by

$$r^{1}_{x_0,x_s} = \begin{cases} 1, \text{ if } x_s \in X_1 \\ 0, \text{ if } x_s \overline{\in} X_1 \end{cases}. \tag{5.10}$$

It follows from (5.10) that for $x_s=0$ the formula $r^{1}_{x_0,0}=1$ is correct, and we may write here $\binom{1}{0}$ in place of unity. For $x_s=1$, we obtain $r^{1}_{x_0,x_1}=1$, which we write as $\binom{1}{1}$. That is, the statement of Theorem 5.2 is true for k=0,1. Suppose that the assertion of the theorem is satisfied for vertices in the subset X_{k-1}, i.e., numbers of paths connecting $x_0 \in X_0$ with the vertices $x_s \in X_{k-1}=\{0,1,\ldots,x_s-0,x_s-1,\ldots,k-1\}$ are given by

$$\left\{\binom{k-1}{0}, \binom{k-1}{1}, \ldots, \binom{k-1}{x_s-0}, \binom{k-1}{x_s-1}, \ldots, \binom{k-1}{k-1}\right\}.$$

Now, from Theorem 5.1 we find for the Pascal triangle graph

$$r^{k}_{x_0,x_s} = r^{k-1}_{x_0,x_s-0} + r^{k-1}_{x_0,x_s-1}, \tag{5.11}$$

and by the induction hypothesis we can substitute $\binom{k-1}{x_0-s}$ for the first term, and $\binom{k-1}{x_s-1}$ for the second, whence

$$r^{k}_{x_0,x_s} = \binom{k-1}{x_s-0} + \binom{k-1}{x_s-1} = \binom{k}{x_s},$$

and so the result is true for k also, and the theorem is proved. The graph of the Pascal triangle with basis <0,1> for k=3 is shown in Figure 56b.

5.3 AN ASYMPTOTIC FORMULA FOR THE NUMBER OF PATHS IN THE GENERALIZED m-ARITHMETIC GRAPH

The method of determining the number of paths connecting $x_0 \in X_0$ with $x_s \in X_k$ by means of the adjacency matrix of the graph G(X,F) is convenient for small k, but involves computational difficulties if k is large. This situation can be avoided if it is possible to find an asymptotic formula to estimate the number of paths connecting x_0 with $x_s \in X_k$ as $k \to \infty$.

From the definition of the generalized m-arithmetic graph with the basis $<\alpha_0,\ldots,\alpha_{m-1}>$ it follows that from each vertex, independent of its value (except terminal vertices), there originate exactly m edges, and each of these corresponds uniquely to one of the basis values. On the other hand, the graph G(X,F) consists of a set of paths of length k of the form $s=(x_0,x_1,\ldots,x_s)$, where $x_0 \in X_0, x_1 \in X_1,\ldots,x_s \in X_k$. As a simple example, take the value of the root to be $x_0=0$, for which we then have

$$x_1 = x_0 + \beta_1; \quad x_2 = x_1 + \beta_2; \quad \ldots; \quad x_s = x_{s-1} + \beta_s,$$

where $\beta_j \in \{\alpha_0,\alpha_1,\ldots,\alpha_{m-1}\}$. With $x_0=0$, we can write $x_s=\beta_1+\beta_2+\cdots+\beta_k$.

Lemma 5.2. In the generalized m-arithmetic triangle with the basis $<\alpha_0,\ldots,\alpha_{m-1}>$, the number of all possible paths of the form s is $N=m^k$.

Proof. The component β_1 of the path s may take on any of the m basis elements as value. The component β_2 may take on the same values independently of β_1, and so on for each $\beta_j \in B$. Using the "product rule" of combinatorial analysis [44], we have

$$N = |\{\beta_1,\beta_2,\ldots,\beta_k\},\ \beta_j \in B| = m^k.$$

In terms of the elements of the adjacency matrix, it follows from the lemma that the number of paths of the form s may be written as

$$\sum_{x_s \in X_k} r^k_{x_0, x_s} = m^k.$$

Then each vertex $x_s \in X_k$ may be assigned the probability

$$p(x_s) = p(x_s = \beta_1 + \beta_2 + \cdots + \beta_k) = \frac{r^k_{x_0, x_s}}{m^k},$$

where

$$\sum_{x_s \in X_k} p(x_s) = 1.$$

The individual terms of the expression $x_s = \beta_1 + \cdots + \beta_k$ may be considered independent random variables taking on as values the basis elements. The expected value of the sum of independent random variables equals the sum of the expected values, and so we have

$$a_k = ka_1 = k\left(\alpha_0 + p\frac{m-1}{2}\right), \tag{5.12}$$

where a_1 is the expected value of $\beta_1 \in X_1$.

The variance in this case is determined by the formula

$$\sigma_k^2 = k\sigma_1^2 + \frac{kp^2(m+1)}{6m}, \tag{5.13}$$

where σ_1 is the variance of $\beta_1 \in X_1$.

Using a known limit theorem [38] from probability theory for $k\to\infty$, and (5.12) and (5.13), we obtain the asymptotic formula for the number of paths connecting $x_0 \in X_0$ with $x_s \in X_k$:

$$r^k_{x_0,x_s} = \frac{m^k}{\sigma_k\sqrt{2\pi}} \exp\left[-\frac{(x_s - a_k)^2}{2\sigma_k^2}\right] + O\left(\frac{m^{k-1}}{\sqrt{k}}\right). \tag{5.14}$$

The result (5.14) could also be used to obtain approximate values of the elements of the generalized Pascal triangle of order m, for large values of k.

5.4 GENERALIZED m-ARITHMETIC GRAPHS AND SPATIAL ISOMORPHISMS

We consider now the spatial representation of the m-arithmetic graph with the basis $<\alpha_0,\ldots,\alpha_{m-1}>$, and a theorem on the isomorphism of this graph with the planar graph.

Definition 5.5. The module-graph of the generalized m-arithmetic graph with the basis $<\alpha_0,\ldots,\alpha_{m-1}>$ is the graph $G_{x_p}(X',F')$ defined in the following way:

(a) $X' = X_0' \cup X_1'$ and $X_0' \cap X_1' = \varnothing$, where $X_0' = \{x_p\}$, $X_1' = \{x_p+\alpha_0, x_p+\alpha_1, \ldots, x_p+\alpha_{m-1}\}$;

(b) $\exists! x_p \in X_0' \left[F_{x_p}' = X_1' \wedge (F')^{-1}x_p = \varnothing\right]$, i.e., the vertex $x_p \in X_0'$ is the root of the graph $G_{x_p}(X',F')$;

(c) $\forall x_s \in X_1' \, [Fx_s = \varnothing]$.

It follows from the definition of the module-graph $G_{x_p}(X',F')$ that this graph is a subgraph of G(X,F), i.e.,

$$G_{x_p}(X',F') \subseteq G(X,F).$$

The module-graph $G_{x_p}(X',F')$ with $x_p=0$ and $X_1'=\{\alpha_0,\alpha_1,\alpha_2\}$ is shown in Figure 57a.

The notion of orientedness of a graph need not be arbitrarily imposed on the edges and vertices of the graph $G_{x_p}(X',F')$ generated here. That is, the edges may have any geometric length and direction without violating the properties of incidence and connectedness. Further, without violating the conditions of Definition 5.5, in the graph $G_{x_p}(X',F')$ we may:

(1) identify the root $x_p \in X_0'$ with the origin of coordinates in m-dimensional space;

(2) take each edge $\vec{J}(x_p,x_i)$, where $x_i \in X_1'$, to have the direction and magnitude corresponding to the unit vectors $\vec{e}_0$, $\vec{e}_1$, ..., $\vec{e}_{m-1}$ of m-dimensional space.

If in fact these identifications are made, we obtain the spatial representation of the module-graph $G_{x_p}(X',F')$, coinciding, up to isomorphisms, with the usual m-dimensional coordinate space. The vertex x_p is at the origin, and the edges $\vec{J}(x_p,x_p+\alpha_0),\ldots,\vec{J}(x_p,x_p+\alpha_{m-1})$ coincide with the unit vectors. Such a representation for $x_p=0$ and $X_1'=\{\alpha_0,\alpha_1,\alpha_2\}$ is shown in Figure 57b.

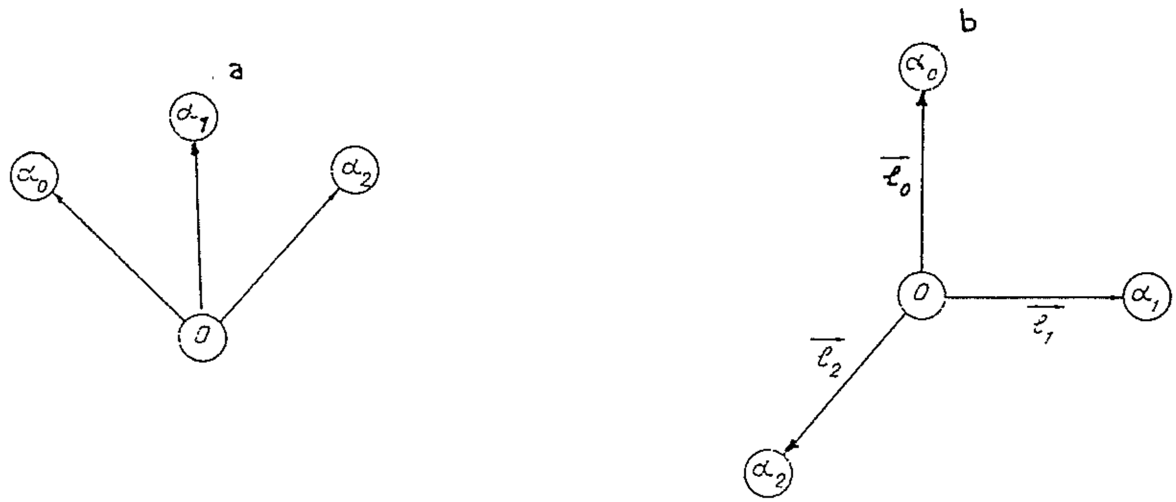

Figure 57

In what follows, when we speak of a module-graph we will be referring to its spatial representation.

Lemma 5.3. The generalized m-arithmetic graph with the usual basis may be represented by means of its module-graphs in m-space, i.e.,

$$G(X,F) = \bigcup_{x_p} G_{x_p}(X', F'),$$

where $x_p \in X_0 \cup X_1 \cup X_2 \cup \ldots \cup X_{k-1}$.

Proof. For k=1, the assertion is easy to show, i.e.,

$$G(X,F) = \bigcup_{x_0} G_{x_p}(X',F') = G_{x_0}(X',F'),$$

where $X_0 = X_0' = \{x_0\}$ and $X_1 = X_1'$.

As for the mappings F and F', F'=F will fulfill the condition. It follows, then, that for k=1 the generalized m-arithmetic graph G(F,X) with the basis $<\alpha_0,\ldots,\alpha_{m-1}>$ coincides, up to isomorphisms, with its module-graph.

Now suppose the assertion of the lemma is valid for the m-arithmetic graph with the set of vertices $X''=X_0 \cup X_1 \cup \cdots \cup X_{k-1}$ (path lengths $k-1$), and denote this graph by $G_{k-1}(X'',F'')$. For this graph, (5.15) has the form

$$G_{k-1}(X'',F'') = \bigcup_{x_p} G_{x_p}(X',F'),$$

where $x_p \in X_0 \cup X_1 \cup \cdots \cup X_{k-2}$.

Then, we will have for the graph G(X,F)

$$G(X,F) = G_{k-1}(X'',F'') \bigcup_{x_p \in X_{k-1}} G_{x_p}(X',F'), \tag{5.16}$$

and from Definition 5.2 and Definition 5.5 it follows that

$$\forall x_p \in X_{k-1}\left[Fx_p = \{x_p+\alpha_0, x_p+\alpha_1, \ldots, x_p+\alpha_{m-1}\} \wedge F'x_p = Fx_p\right].$$

And for the subset of vertices X_k, we have that

$$X_k = \bigcup_{x_p \in X_{k-1}} F'x_p,$$

according to which, if $G_{k-1}(X'',F'')$ may be represented by its module-graph, then for G(X,F) in (5.16) it follows that

$$G(X,F) = \bigcup_{x_p \in Q} G_{x_p}(X',F') \bigcup_{x_p \in X_{k-1}} G_{x_p}(X',F') = \bigcup_{x_p \in D} G_{x_p}(X',F'),$$

where $Q=X_0 \cup X_1, \ldots, X_{k-2}$ and $D=X_0 \cup X_1 \cup \cdots \cup X_{k-1}$, which is the assertion of the lemma.

Definition 5.6. The graph obtained as the union of module-graphs by (5.15) is said to be the spatial representation of the generalized m-arithmetic graph with the basis $<\alpha_0,\ldots,\alpha_{m-1}>$, or the spatial arithmetic graph, and is denoted by $G_S(X,F)$.

Figure 58 shows the spatial arithmetic graph with the basis $<1,3,5>$ and k=3; the vertices are shown as points, and the edges are not oriented.

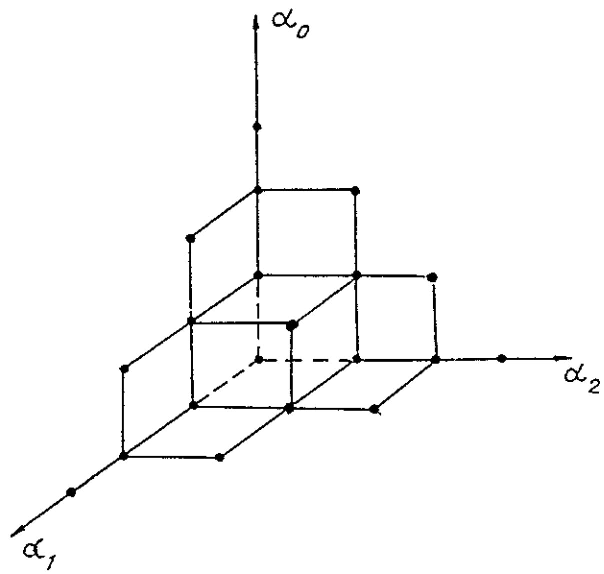

Figure 58

Theorem 5.3. The spatial arithmetic graph $G_S(X,F)$ is isomorphic to the generalized m-arithmetic graph G(X,F).

Proof. The spatial graph may be obtained as the union of the module-graphs $G_{x_p}(X',F')$, the edges of which have defined directions, and coincide with the direction and magnitude of the unit vectors. This last condition does not violate the properties of incidence and connectedness of the vertices of G(X,F), and by a known theorem [32] this is a necessary and sufficient condition for the isomorphism of G(X,F) and $G_S(X,F)$.

5.5 PROPERTIES OF THE CROSS SECTIONS OF THE SPATIAL ARITHMETIC GRAPH

We discuss now some properties of the cross sections of the spatial m-arithmetic graph, from which we might derive some general methods for obtaining recurrent sequences from various analogs of the Pascal triangle.

For each vertex $x_s \in X_k$ of the spatial arithmetic graph $G_S(X,F)$, we have the valid representation $x_s = x_0 + \beta_1 + \beta_2 + \cdots + \beta_k$, where $\beta_j \in \{\alpha_0, \cdots, \alpha_{m-1}\}$, and if we take $x_0 = 0$, then $x_s = \beta_1 + \cdots + \beta_k$. On the other hand, we can also represent x_s by the linear expression

$$x_s = \mu_0 \alpha_0 + \mu_1 \alpha_1 + \cdots + \mu_{m-1} \alpha_{m-1},$$

where μ_i is the number of components β_j taking on the value α_i.

Definition 5.7. The numbers $\mu_0, \mu_1, \cdots, \mu_{m-1}$ are said to be the coordinates of the vertex $x_s \in X_k$.

The number of paths connecting $x_0 \in X_0$ with $x_s \in X_k$, i.e., $r^k_{x_0,x_s}$, may be determined by the multinomial formula

$$r^k_{x_0,x_s} = \frac{(\mu_0 + \mu_1 + \cdots + \mu_{m-1})!}{\mu_0! \mu_1! \cdots \mu_{m-1}!}, \tag{5.18}$$

where $\mu_0 + \cdots + \mu_{m-1} = k$.

Definition 5.8. By a cross section of the spatial arithmetic graph $G_S(X,F)$, we will mean the sum of the numbers of paths connecting the origin of coordinates $x_0 \in X_0$ with the vertices x_s of this graph which lie in the hyperplane

$$a_0 y_0 + a_1 y_1 + \cdots + a_{m-1} y_{m-1} = n \tag{5.19}$$

where the a_i are positive whole numbers.

Denote this cross section by T_n. Then, since the coordinates $(\mu_0,\ldots,\mu_{m-1})$ of the vertices $x_s \in X$ of the graph $G_S(X,F)$ must satisfy (5.19), i.e.,

$$a_0\mu_0 + a_1\mu_1 + \cdots + a_{m-1}\mu_{m-1} \equiv n, \tag{5.20}$$

T_n is determined by the equation

$$T_n = \sum_{x_s \in X} \frac{(\mu_0+\mu_1+\cdots+\mu_{m-1})!}{\mu_0!\mu_1!\cdots\mu_{m-1}!}. \tag{5.21}$$

Theorem 5.4. The cross section T_n of the graph $G_S(X,F)$ satisfies the recurrence formula

$$T_n = T_{n-a_0} + T_{n-a_1} + \cdots + T_{n-a_{m-1}}, \tag{5.22}$$

where $T_0=1$ and $T_k=0$, if $k<0$.

Proof. Subtracting a_0 from each side of (5.19) we have

$$a_0(y_0-1) + a_1y_1 + \cdots + a_{m-1}y_{m-1} = n-a_0.$$

From (5.20) and (5.21), the cross section T_{n-a_0} is given by

$$T_{n-a_0} = \sum_{x_s \in X} \frac{(\mu_0+\mu_1+\cdots+\mu_{m-1}-1)!}{(\mu_0-1)!\mu_1!\cdots\mu_{m-1}!}.$$

Likewise, for a_1 we will have

$$T_{n-a_1} = \sum_{x_s \in X} \frac{(\mu_0+\mu_1+\cdots+\mu_{m-1}-1)!}{\mu_0!(\mu_1-1)!\cdots\mu_{m-1}!},$$

and so on, up to a_{m-1}:

$$T_{n-a_{m-1}} = \sum_{x_s \in X} \frac{(\mu_0+\mu_1+\cdots+\mu_{m-1}-1)!}{\mu_0!\mu_1!\cdots(\mu_{m-1}-1)!}.$$

Substituting these expressions for the T's on the right side of (5.22), we have

$$\sum_{x_s \in X} \left[\frac{(\mu_0+\mu_1+\cdots+\mu_{m-1}-1)!}{(\mu_0-1)!\mu_1!\cdots\mu_{m-1}!} + \frac{(\mu_0+\mu_1+\cdots+\mu_{m-1}-1)!}{\mu_0!(\mu_1-1)!\cdots\mu_{m-1}!} + \right.$$
$$\left. \cdots + \frac{(\mu_0+\mu_1+\cdots+\mu_{m-1}-1)!}{\mu_0!\mu_1!\cdots(\mu_{m-1}-1)!} \right]. \tag{5.23}$$

But after an elementary transformation, (5.23) may be written in the form

$$\sum_{x_s \in X} \frac{(\mu_0+\mu_1+\cdots+\mu_{m-1})!}{\mu_0!\mu_1!\cdots\mu_{m-1}!},$$

which is just T_n, and this completes the proof.

For $n=0$, the plane (5.19) passes through the origin of coordinates; in this case we take $T_0=1$. For values of $n<0$, $T_n=0$, since $G_S(X,F)$ has no vertices in the negative half-space. Theorem 5.4 says, in effect, that among the cross sections by parallel planes we have the recurrence relation (5.22). Thus, we can think of the spatial graph of the generalized Pascal triangle of order m as the source of an infinite number of recurrence relations of the type (5.22).