CHAPTER 6

MATRICES AND DETERMINANTS OF BINOMIAL AND GENERALIZED BINOMIAL COEFFICIENTS AND OTHER NUMBERS

Matrices and determinants composed of binomial and generalized binomial coefficients, and Fibonacci, Lucas, and Catalan numbers arise in the solution of specific systems of algebraic and difference equations, and may have interesting properties. The problems of evaluating determinants composed of binomial coefficients arranged in some specific form in the Pascal triangle have long been known; some of these may be found in [292]. Material on the construction of matrices and determinants of elements of the Pascal triangle, its generalizations, and other numbers, may be found in the papers of M. Bicknell and V.E. Hoggatt [72, 74, 75, 204-206], and in [92, 93, 105, 150, 258, 269, 282, 285, 326, 333, 342, 375].

6.1 MATRICES AND DETERMINANTS OF ELEMENTS OF THE PASCAL TRIANGLE

It turns out that there are many ways of choosing a square matrix of elements of the Pascal triangle so that the determinant is unity, or may be evaluated in terms of the corresponding binomial coefficients by some explicit formula. In [74] there are specific examples of choices of a square matrix from the Pascal triangle so that the determinant is unity.

With the Pascal triangle written in rectangular form, let us choose from it the $n \times n$ matrix $A=(a_{ik})$ as shown,

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & . & . & . \\ 1 & 2 & 3 & 4 & 5 & . & . & . \\ 1 & 3 & 6 & 10 & 15 & . & . & . \\ 1 & 4 & 10 & 20 & 35 & . & . & . \\ 1 & 5 & 15 & 35 & 70 & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}_{n\times n},$$

where a_{ik} is the i^{th} row, k^{th} column element, $1\leq i,k\leq n$. It is not difficult to show that

$$a_{ik} = \binom{i+k-2}{k-1}.$$

In [74] it is shown that the determinant of any submatrix consisting of the first m rows and columns of A has a value of one; for example

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} = 1.$$

With the Pascal triangle in right triangular form, let us choose from it the $n\times n$ matrix B as shown,

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 1 & 1 & 0 & 0 & 0 & . & . & . \\ 1 & 2 & 1 & 0 & 0 & . & . & . \\ 1 & 3 & 3 & 1 & 0 & . & . & . \\ 1 & 4 & 6 & 4 & 1 & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}_{n\times n}, \text{ where } b_{ik} = \binom{i-1}{k-1}.$$

Again in [74], it is shown that the determinant of any submatrix of B which contains the column of 1's, has value one; for example

$$\begin{vmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \\ 1 & 5 & 10 & 10 \end{vmatrix} = 1.$$

Along with the matrices A and B in [74], there are also some other matrices, the determinants of which have simple evaluations. Thus, let R(m,r) be the $m \times m$ submatrix of A which has as its first row the second row of A, and as its first column the r^{th} column of A; for example,

$$R(4,3) = \begin{bmatrix} 3 & 4 & 5 & 6 \\ 6 & 10 & 15 & 21 \\ 10 & 20 & 35 & 56 \\ 15 & 35 & 70 & 126 \end{bmatrix}.$$

The authors show that the determinant of R(m,r) equals the binomial coefficient $\binom{m+r-1}{m}$, and also satisfies the recurrence formula

$$\det R(m,r) = \det R(m-1,r) + \det R(m,r-1).$$

For some specific values of m,r we find that

$$\det R(m,1) = 1, \quad \det R(1,r) = r, \quad \det R(m,2) = m+1,$$

$$\det R(2,r) = \frac{1}{2}r(r+1), \quad \det R(4,3) = 15.$$

From Netto's book on combinatorics [292], and omitting the proofs, here are some interesting determinants D_k composed of binomial coefficients.

$$D_1 = \begin{vmatrix} \binom{n}{m} & \binom{n}{m+1} & \cdots & \binom{n}{m+r} \\ \binom{n+1}{m} & \binom{n+1}{m+1} & \cdots & \binom{n+1}{m+r} \\ \vdots & \vdots & & \vdots \\ \binom{n+r}{m} & \binom{n+r}{m+1} & \cdots & \binom{n+r}{m+r} \end{vmatrix}$$

$$= \frac{\binom{n}{m}\binom{n+1}{m}\cdots\binom{n+r}{m}}{\binom{m+r}{r}\binom{m+r-1}{r-1}\cdots\binom{m+1}{1}}, \quad m \le n,$$

$$D_2 = \begin{vmatrix} \binom{n}{m} & \binom{n}{m+2} & \cdots & \binom{n}{m+2r} \\ \binom{n+1}{m} & \binom{n+1}{m+2} & \cdots & \binom{n+1}{m+2r} \\ \vdots & \vdots & & \vdots \\ \binom{n+r}{m} & \binom{n+r}{m+2} & \cdots & \binom{n+r}{m+2r} \end{vmatrix}$$

$$= 2^{\frac{1}{2}r(r-1)} \frac{\binom{n}{m+r}\binom{n+1}{m+r-1}\cdots\binom{n+r}{m}}{\binom{m+2r}{r}\binom{m+2r-2}{r-2}\cdots\binom{m+2}{1}}, \quad m \le n-r,$$

$$D_3 = \begin{vmatrix} \binom{n}{m} & \binom{n}{m+3} & \cdots & \binom{n}{m+3r} \\ \binom{n+1}{m} & \binom{n+1}{m+3} & \cdots & \binom{n+1}{m+3r} \\ \vdots & \vdots & & \vdots \\ \binom{n+r}{m} & \binom{n+r}{m+3} & \cdots & \binom{n+r}{m+3r} \end{vmatrix}$$

$$= 3^{\frac{1}{2}r(r+1)} \frac{\binom{n}{m+2r}\binom{n+1}{m+2r-2}\cdots\binom{n+r}{m}}{\binom{m+3r}{r}\binom{m+3r-3}{r-1}\cdots\binom{m+3}{1}}, \quad m \le n-2r.$$

The determinants D_1, D_2, D_3 may be extended to the case of any whole number k, the result being

$$D_k = \begin{vmatrix} \binom{n}{m} & \binom{n}{m+k} & \cdots & \binom{n}{m+kr} \\ \binom{n+1}{m} & \binom{n+1}{m+k} & \cdots & \binom{n+1}{m+kr} \\ \vdots & \vdots & & \vdots \\ \binom{n+r}{m} & \binom{n+r}{m+k} & \cdots & \binom{n+r}{m+kr} \end{vmatrix}$$

$$= k^{\frac{1}{2}r[r-(-1)^k]} \frac{\binom{n}{m+(k-1)r}\binom{n+1}{m+(k-1)(r-1}\cdots\binom{n+r}{m}}{\binom{m+kr}{r}\binom{m+k(r-1)}{r-1}\cdots\binom{m+k}{1}}, \quad m \le n-(k-1)r.$$

Matrices and determinants composed of binomial coefficients are also discussed in [61, 72, 74, 150, 269, 285].

6.2 MATRICES AND DETERMINANTS OF ELEMENTS OF GENERALIZED PASCAL TRIANGLES

Consider the $n \times n$ matrix A_3 composed of trinomial coefficients, of the form

$$A_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & . & . & . \\ 0 & 1 & 2 & 3 & 4 & . & . & . \\ 0 & 1 & 3 & 6 & 10 & . & . & . \\ 0 & 0 & 2 & 7 & 16 & . & . & . \\ 0 & 0 & 1 & 6 & 19 & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}_{n\times n} .$$

It is not hard to show that A_3 may be represented in the form of a product of matrices composed of binomial coefficients

$$A_3 = F_1 A^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 1 & 0 & 0 & . & . & . \\ 0 & 0 & 2 & 1 & 0 & . & . & . \\ 0 & 0 & 1 & 3 & 1 & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}_{n\times n} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & . & . & . \\ 0 & 1 & 2 & 3 & 4 & . & . & . \\ 0 & 0 & 1 & 3 & 6 & . & . & . \\ 0 & 0 & 0 & 1 & 4 & . & . & . \\ 0 & 0 & 0 & 0 & 1 & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}_{n\times n} .$$

In [74, 75] it is shown that, in general, the matrix A_s of generalized binomial coefficients $\binom{n}{m}_s$ may also be represented as a product $A_s = F_{s-2} A^T$. Using this result, the authors prove that:

(1) the determinant of the $n \times n$ matrix A_s' whose first row and first column coincide with those of A_s, has the value one;

(2) the determinant of the n×n matrix A_s'' whose first column is the first column of A_s, and whose first row is the r^{th} row of A_s, has the value $\binom{n+r-1}{n}$.

Suppose we form the matrix of coefficients in the expansions of the elements in the sequence 1, 1+x, (1+x)(2+x), (1+x)(2+x)(3+x), ...:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 1 & 1 & 0 & 0 & 0 & . & . & . \\ 2 & 3 & 1 & 0 & 0 & . & . & . \\ 6 & 11 & 6 & 1 & 0 & . & . & . \\ 24 & 50 & 35 & 10 & 1 & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}_{n \times n},$$

where the coefficients $a_{r,s}$ are determined by the recurrence

$$a_{n,m} = n a_{n-1,m} + a_{n-1,m-1}, \quad a_{0,0} = 1.$$

Now form from the elements of C, the n^{th} order determinant

$$D_{n,m} = \begin{bmatrix} a_{n,0} & a_{n,1} & \cdots & a_{n,m-1} \\ a_{n+1,0} & a_{n+1,1} & \cdots & a_{n+1,m-1} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+m-1,0} & a_{n+m-1,1} & \cdots & a_{n+m-1,m-1} \end{bmatrix}.$$

Then, according to [333],

$$D_{n,m} = (n!)^m.$$

6.3 MATRICES AND DETERMINANTS OF ELEMENTS OF OTHER ARITHMETIC TRIANGLES

As discussed in Section 1.4, arithmetic triangles composed of Gaussian binomial coefficients, or Fibonacci, Lucas, Stirling, Euler, and other special numbers may form lower triangular, upper triangular, or other kinds of matrices. Studies of such matrices and their applications may be found in [92, 93, 204-206, 258, 282, 326]. We mention a few of these results.

D.A. Lind [258] formed the matrix of Fibonomial coefficients

$$\begin{bmatrix} m \\ r \end{bmatrix} = F_m F_{m-1} \cdots F_{m-r+1} / (F_1, F_2 \cdots F_r), \; r \geq 1,$$

where $\begin{bmatrix} m \\ 0 \end{bmatrix} = 1$, $\begin{bmatrix} m \\ r \end{bmatrix} = 0$ for $0 \leq m \leq r-1$ or $r < 0$. Let $D_{n,k}$ be the determinant of the $n \times n$ matrix (a_{rs}), where

$$a_{rs} = -(-1)^{(s-r+1)(s-r+2)/2} \begin{bmatrix} k+1 \\ s-r+1 \end{bmatrix}, \quad r,s = 1,2,\ldots,n.$$

Then

$$D_{n,k} = \begin{bmatrix} n+k \\ k \end{bmatrix}.$$

We also note the values of some other determinants given in [258]; thus

$$\begin{bmatrix} 1 & 1 & 0 & 0 & . & . & . \\ -1 & 1 & 1 & 0 & . & . & . \\ 0 & -1 & 1 & 1 & . & . & . \\ 0 & 0 & -1 & 1 & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}_{n\times n} = F_{n+1}, \qquad \begin{bmatrix} 2 & -1 & 0 & 0 & . & . & . \\ -1 & 2 & -1 & 0 & . & . & . \\ 0 & -1 & 2 & -1 & . & . & . \\ 0 & 0 & -1 & 2 & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}_{n\times n} = n+1,$$

$$\begin{bmatrix} 2 & 2 & -1 & 0 & . & . & . \\ -1 & 2 & 2 & -1 & . & . & . \\ 0 & -1 & 2 & 2 & . & . & . \\ 0 & 0 & -1 & 2 & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}_{n\times n} = F_{n+1}F_{n+2}, \qquad \begin{bmatrix} 3 & -3 & 1 & 0 & . & . & . \\ -1 & 3 & -3 & 0 & . & . & . \\ 0 & -1 & 3 & -3 & . & . & . \\ 0 & 0 & -1 & 3 & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}_{n\times n} = \frac{1}{2}(n+1)(n+2).$$

T.A. Brennan in [92, 93] studied the properties of the $n\times n$ matrix P_n, formed from a version of the Pascal triangle,

$$P_n = \begin{bmatrix} . & . & . & 0 & 0 & 0 & 1 \\ . & . & . & 0 & 0 & 1 & 1 \\ . & . & . & 0 & 1 & 2 & 1 \\ . & . & . & . & . & . & . \end{bmatrix}_{n\times n}.$$

He proved that the characteristic polynomial is given by

$$|xJ-Q_n| = \sum_{r=0}^{n+1} (-1)^{\frac{r(r+1)}{2}} \begin{bmatrix} n+1 \\ r \end{bmatrix} x^{n-r+1},$$

where Q_n is the matrix obtained from P_n by transposing P_n about its secondary diagonal, and $\begin{bmatrix} n \\ r \end{bmatrix}$ is the Fibonomial coefficient mentioned earlier.