EXPLORING FIBONACCI NUMBERS

Brother Alfred Brousseau
St. Mary's College, Ca.

What are currently known as Fibonacci numbers came into existence as part of a mathematical puzzle problem proposed by Leonardo Pisano (also known as Fibonacci) in his famous book on arithmetic, the Liber Abaci (1202). He set up the following situation for the breeding of rabbits.

Suppose that there is one pair of rabbits in an enclosure in the month of January; that these rabbits will breed another pair of rabbits in the month of February; that pairs of rabbits always breed in the second month following birth and thereafter produce one pair of rabbits monthly. What is the number of pairs of rabbits at the end of December?

To solve this problem, let us set up a table with columns as follows:

(1) Number of pairs of breeding rabbits at the beginning of the given month;

(2) Number of pairs of non-breeding rabbits at the beginning of the month;

(3) Number of pairs of rabbits bred during the month;

(4) Number of pairs of rabbits at the end of the month.

MONTH	(1)	(2)	(3)	(4)
January	1	0	1	2
February	1	1	1	3
March	2	1	2	5
April	3	2	3	8
May	5	3	5	13
June	8	5	8	21
July	13	8	13	34
August	21	13	21	55
September	34	21	34	89
October	55	34	55	144
November	89	55	89	233
December	144	89	144	377

The answer to the original question is that there are 377 pairs of rabbits at the end of December. But the curious fact that characterizes the series of numbers evolved in this way is: any one number is the sum of the two previous numbers. Furthermore, it will be observed that all four columns in the above table are formed from numbers of the same series which has since come to be known as THE Fibonacci series: 0, 1, 1, 2, 3, 5, 8, 13, 21, ...

1. EXPLORATION

Did anybody ever find out what happened to the "Fibonacci rabbits" when they began to die? Since they have been operating with such mathematical regularity in other respects, let us assume the following as well. A pair of rabbits that is bred in February of one year breeds in April and every month thereafter including February of the following year. Then this pair of rabbits dies at the end of February.

(1) How many pairs of rabbits are there at the end of December of the second year?

(2) How many pairs of rabbits would there be at the end of n months, where n is greater than or equal to 12? (See what follows for notation.)

Assume that the original pair of rabbits dies at the end of December of the first year.

2. NAMES FOR ALL FIBONACCI NUMBERS

The inveterate Fibonacci addict tends to attribute a certain individuality to each Fibonacci number. Mention 13 and he thinks F_7; 55 and F_{10} flashes through his mind. But regardless of this psychological quirk, it is convenient to give the Fibonacci numbers identification tags and since they are infinitely numerous, these tags take the form of subscripts attached to the letter F. Thus 0 is denoted F_0; the first 1 in the series is F_1; the second 1 is F_2; 2 is F_3; 3 is F_4; 5 is F_5; etc. The following table for F_n shows a few of the Fibonacci numbers and then provides additional landmarks so that it will be convenient for each Fibonacci explorer to make up his own table.

n	F_n	n	F_n
0	0	11	89
1	1	12	144
2	1	13	233
3	2	14	377
4	3	20	6765
5	5	30	832040
6	8	40	102334155
7	13	50	12586269025
8	21	60	1548008755920
9	34	70	190392490709135
10	55	80	23416728348467685

3. SUMMATION PROBLEMS

The first question we might ask is: What is the sum of the first n terms of the series? A simple procedure for answering this question is to make up a table in which we list the Fibonacci numbers in one column and their sum up to a given point in another.

n	F_n	Sum
1	1	1
2	1	2
3	2	4
4	3	7
5	5	12
6	8	20
7	13	33
8	21	54

What does the sum look like? It is not a Fibonacci number, but if we add 1 to the sum, it is the Fibonacci number two steps ahead. Thus we could write:

$$1 + 2 + 3 + \ldots + 34 + 55\ (F_{10}) = 143 = 144 - 1 = (F_{12}) - 1,$$

where we have indicated the names of the key Fibonacci numbers in parentheses. It is convenient at this point to introduce the summation notation. The above can be written more concisely:

$$\sum_{k=1}^{10} F_k = F_{12} - 1.$$

The Greek letter Σ (sigma) means: Take the sum of quantities F_k, where k runs from 1 to 10. We shall use this notation in what follows.

It appears that the sum of any number of consecutive Fibonacci numbers starting with F_1 is found by taking the Fibonacci number two steps beyond the last one in the sum and subtracting 1. Thus if we were to add the first hundred Fibonacci numbers together we would expect to obtain for an answer $F_{102} - 1$. Can we be sure of this? Not completely, unless we have provided some form of proof. We shall begin with a numerical proof meaning a proof that uses specific numbers. The line of reasoning employed can then be readily extended to the general case.

Let us go back then to the sum of the first ten Fibonacci numbers. We have seen that this sum is $F_{12} - 1$. Now suppose that we add 89 (or F_{11}) to both sides of the equation. Then on the lefthand side we have the sum of the first eleven Fibonacci numbers and on the right we have

$$144 - 1 + 89 = F_{12} - 1 + F_{11} = 233 - 1 = F_{13} - 1$$

Thus, proceeding from the sum of the first ten Fibonacci numbers to the sum of the first eleven Fibonacci numbers, we have shown that the same type of relation must hold. Is it not evident that we could now go on from eleven to twelve; then from twelve to thirteen; etc., so that the relation must hold in general?

This is the type of reasoning that is used in the general proof by mathematical induction. We suppose that the sum of the first n Fibonacci numbers is $F_{n+2} - 1$. In symbols:

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1$$

We add F_{n+1} to both sides and obtain

$$\sum_{k=1}^{n+1} F_k = F_{n+2} - 1 + F_{n+1} = F_{n+3} - 1$$

by reason of the fundamental property of Fibonacci series that the sum of any two consecutive Fibonacci numbers is the next Fibonacci number. We have now shown that if the summation holds for n, it holds also for n + 1. All that remains to be done is to go back to the beginning of the series and draw a complete conclusion. Let us suppose, as can readily be done, that the formula for the sum of the first n terms of the Fibonacci sequence holds for $n \leq 7$. Since the formula holds for seven, it holds for eight; since it holds for eight, it holds for nine; etc., etc. Thus the formula is true for all integral positive values of n.

We have seen from this example that there are two parts to our mathematical exploration. In the first we observe and thus arrive at a formula. In the second we prove that the formula is true in general.

Let us take one more example. Suppose we wish to find the sum of all the odd-numbered Fibonacci numbers. Again, we can form our table.

n	F_n	Sum
1	1	1
3	2	3
5	5	8
7	13	21
9	34	55
11	89	144

This is really too easy. We have come up with a Fibonacci number as the sum. Actually it is the very next after the last quantity added. We shall leave the proof to the explorer. However, the question of fitting the above results into notation might cause some trouble. What we need is a type of subscript that will give us just the odd numbers and no others. For the above sum to 11, we would write

$$\sum_{k=1}^{6} F_{2k-1} = F_{12}$$

It will be seen that when k is 1, 2k-1 is 1; when k is 2, 2k-1 is 3; etc., and when k is 6, 2k-1 is 11. In general, the relation for the sum of the first n odd-subscripted Fibonacci numbers would be:

$$\sum_{k=1}^{n} F_{2k-1} = F_{2n}.$$

4. PROBLEMS FOR EXPLORATION

1. Determine the sum of the first n even-subscripted Fibonacci numbers.

2. If we take every fourth Fibonacci number and add, four series are possible:

   (a) Subscripts 1, 5, 9, 13, ...
   (b) Subscripts 2, 6, 10, 14, ...
   (c) Subscripts 3, 7, 11, 15, ...
   (d) Subscripts 4, 8, 12, 16, ...

Determine the sum of the first n terms in each of these series. Hint: Look for products or squares or near-products or near-squares of Fibonacci numbers as the result.

3. If we take every third Fibonacci number and add, three series are possible:

   (a) Subscripts 1, 4, 7, 10, ...
   (b) Subscripts 2, 5, 8, 11, ...
   (c) Subscripts 3, 6, 9, 12, ...

Find the sum of the first n terms in each of those series. Hint: Double the sum and see whether you are near a Fibonacci number.

***** ***** *****

RESEARCH PROJECT: FIBONACCI NIM

Consider a game involving two players in which initially there is a group of 100 or less objects. The first player may reduce the pile by any Fibonacci number. The second does likewise. The player who makes the last move wins the game. Can the first player always win the game?