A PRIMER FOR THE FIBONACCI NUMBERS: PART IV

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FIBONACCI AND LUCAS VECTORS

1. INTRODUCTION

In the primer, Part III, it was noted that if $V = (x, y)$ is a two-dimensional vector and A is a 2×2 matrix, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $V' = AV$ is a two-dimensional vector, $V' = (x', y') = (ax + by, cx + dy)$. Here, V and consequently V', are expressed as column vectors. The matrix A is said to transform, or map, the vector V onto the vector V'. The matrix A is called the mapping matrix or transformation matrix.

2. SOME MAPPING MATRICES

The zero matrix, $Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, maps every vector V onto the zero vector $\emptyset = (0, 0)$. The identity matrix, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, maps every vector V onto itself; that is, $IV = V$. For any real number k, the matrix $B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ maps vectors $V = (k, -k)$ onto the zero vector $\emptyset$. Such a mapping as determined by B is called a many-to-one mapping.

If the only vector mapped onto $\emptyset$ is the vector $\emptyset$ itself, the mapping is a one-to-one mapping. A matrix A determines a one-to-one mapping of two-dimensional vectors onto two-dimensional vectors if, and only if, $\det A \neq 0$. If $\det A \neq 0$, for each vector U, there exists a vector V such that $AV = U$. Note, however, that for matrix B above, $B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ 2x + 2y \end{pmatrix}$. There is no vector V such that $BV = (0, 1)$.

3. GEOMETRIC INTERPRETATIONS OF 2x2 MATRICES AND 2-DIMENSIONAL VECTORS

As in Primer III, the vector $V = (x, y)$ is interpreted as a point in a rectangular coordinate system. Thus the geometric concepts of length, direction, slope and angle are associated with the vector V.

A non-zero scalar multiple of the identity matrix, kI, maps the vector $U = (a, b)$ onto the vector $V = (ka, kb)$. The length of V, $|V|$, is equal to

$|k||U|$. There is no change in slope but if $k < 0$ the sense or direction is reversed.

The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ maps a vector onto the reflection vector with respect to the line through the origin with slope one. Note that different vectors may be rotated through different angles!

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ preserves the first component of a vector while annihilating the second component. Every vector is mapped onto a vector on the X-axis.

The matrix $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ rotates all vectors through the same angle θ (theta), in a counterclockwise direction if theta is a positive angle. There is no change in length. This seems to contradict the notion of a matrix having vectors whose slopes are not changed, but in this case, the characteristic values are complex; thus, there are no real characteristic vectors.

4. THE CHARACTERISTIC VECTORS OF THE Q-MATRIX

The Q matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ does not generally preserve the length of a vector $U = (x, y)$. Also, different vectors are in general rotated through different angles.

The characteristic equation of the Q matrix is

$$\lambda^2 - \lambda - 1 = 0$$

with roots $\lambda_1 = (1 + \sqrt{5})/2$ and $\lambda_2 = (1 - \sqrt{5})/2$, which are the characteristic roots, or eigenvalues, for Q.

To solve for a pair of corresponding characteristic vectors consider

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix} , \qquad x^2 + y^2 \neq 0 .$$

Then

$$(1 - \lambda)x + y = 0 .$$

Thus, a pair of characteristic vectors are X_1 and X_2 with slopes m_1 and m_2,

$$X_1 = (\lambda_1 x, x) , \quad |X_1| \neq 0 , \qquad m_1 = (\sqrt{5} - 1)/2 ,$$

$$X_2 = (\lambda_2 x, x), \quad |X_2| \neq 0, \qquad m_2 = -(\sqrt{5} + 1)/2 .$$

What happens when the matrix Q^2 is applied to the characteristic vectors X_1 and X_2 of matrix Q? Since

$$Q^2X_1 = Q(QX_1) = Q(\lambda X_1) = \lambda QX_1 = \lambda^2 X_1 ,$$

clearly X_1 is a characteristic vector of the matrix Q^2 as well as a characteristic vector of matrix Q. The characteristic roots of Q^2 are the squares of the characteristic roots of matrix Q. In general, if λ_1 and λ_2 are the characteristic roots of Q, then λ_1^n and λ_2^n are the characteristic roots of Q^n. But the characteristic equation for Q^n is

$$0 = \lambda^2 - (F_{n+1} + F_{n-1})\lambda + (F_{n+1}F_{n-1} - F_n^2) = \lambda^2 - L_n\lambda + (-1)^n ,$$

recalling that $L_n = F_{n+1} + F_{n-1}$ and that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

Applying the known identity $L_n^2 = 5F_n^2 + 4(-1)^n$, it follows that

$$\lambda_1^n = [(1 + \sqrt{5})/2]^n = (L_n + \sqrt{5}F_n)/2 \text{ and } \lambda_2^n = [(1 - \sqrt{5})/2]^n = (L_n - \sqrt{5}F_n)/2 .$$

5. FIBONACCI AND LUCAS VECTORS AND THE Q MATRIX

Let $U_n = (F_{n+1}, F_n)$ and $V = (L_{n+1}, L_n)$ be denoted as Fibonacci and Lucas vectors, respectively. We note that

$$|U_n|^2 = F_{n+1}^2 + F_n^2 = F_{2n+1} ,$$

$$|V_n|^2 = L_{n+1}^2 + L_n^2 = (5F_{n+1}^2 + (-1)^{n+1}4) + (5F_n^2 + (-1)^n 4) = 5F_{2n+1} .$$

It is well-known that the slopes of the vectors U_n and V_n (the ratios F_n/F_{n+1} and L_n/L_{n+1}) approach the slope $(\sqrt{5} - 1)/2$ of the characteristic vector X_1.

Since $Q^mQ^n = Q^{m+n}$, it is easy to verify that

$$F_{m+1}F_{n+1} + F_mF_n = F_{m+n+1}$$

by equating elements in the upper left in the above matrix equation. In a similar manner it follows that

$$F_{m+1}F_{n+2} + F_mF_{n+1} = F_{m+n+2} ,$$

$$F_{m+1}F_n + F_mF_{n-1} = F_{m+n} .$$

Adding these two equations and using $L_{n+1} = F_{n+2} + F_n$ it follows that

$$F_{m+1}L_{n+1} + F_mL_n = L_{m+n+1} \quad .$$

From the above identities it is easy to verify that

$$Q^{n+1}V_0 = QV_n = V_{n+1} \quad ,$$

$$Q^{n+1}U_0 = QU_n = U_{n+1} \quad ,$$

$$Q^nV_m = V_{m+n+1} \quad ,$$

$$Q^nU_m = U_{m+n+1} \quad .$$

6. A SPECIAL MATRIX

Let $P = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$; then from

$$L_{n+1} = F_{n+1} + 2F_n \quad , \qquad L_n = 2F_{n+1} - F_n \quad ,$$

$$5F_{n+1} = L_{n+1} + 2L_n \quad , \qquad 5F_n = 2L_{n+1} - L_n \quad ,$$

it follows that

$$PU_n = (F_{n+1} + 2F_n, \;\; 2F_{n+1} - F_n) = V_n$$

$$PV_n = (L_{n+1} + 2L_n, \;\; 2L_{n+1} - L_n) = 5U_n$$

Also

$$PQ^n = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix}$$

$$P^2Q^n = 5Q^n$$

Notice that $\det(PQ^n) = (\det P)(\det Q^n) = 5(-1)^{n+1} = L_{n+1}L_{n-1} - L_n^2$.

We now discuss two geometric properties of matrix P. Let $U = (x, y)$, $|U|^2 = x^2 + y^2 \neq 0$. Now, $PU = (x + 2y, 2x - y)$ and $|PU|^2 = 5(x^2 + y^2) = 5|U|^2$; thus matrix P magnifies each vector length by $\sqrt{5}$.

If $\tan \alpha = y/x$, we say $\alpha = \mathrm{Tan}^{-1}\, y/x$, read "$\alpha$ is an angle whose tangent is y/x." Let $\tan \alpha = y/x$ and $\tan \beta = (2x - y)/(x + 2y)$. From the identity $\tan(\alpha + \beta) = (\tan \alpha + \tan \beta)/(1 - \tan \alpha \tan \beta)$ we may now see what effect P has on the slope of vector $U = (x, y)$.

Now, recalling that $x^2 + y^2 \neq 0$,

$$\tan(\alpha+\beta) = \tan\left(\text{Tan}^{-1}\frac{y}{x} + \text{Tan}^{-1}\frac{2x-y}{x+2y}\right) = \frac{2(x^2+y^2)}{x^2+y^2} = 2 .$$

What does this mean? Consider two vectors A and B, the first inclined at an angle α with the positive X-axis and the second inclined at an angle β with the positive X-axis when the angles are measured positively in the counter-clockwise direction. The angle bisector ψ of the angle between vectors A and B is such that $\alpha - \psi = \psi - \beta$ whether or not α is greater than β or the other way around. Solving for ψ yields

$$\psi = (\alpha + \beta)/2 .$$

Thus ψ is the arithmetic average of α and β. Also we note that $\alpha + \beta = 2\psi$. The tangent of double the angle is given by $\tan 2\psi = (2\tan\psi)/(1 - \tan^2\psi)$. If we let $\tan\psi = (\sqrt{5} - 1)/2$, then it is an easy exercise in algebra to find that $\tan 2\psi = 2$. But, $\tan(\alpha+\beta) = 2$; therefore, we would like to conclude that the angle bisector between vectors U and PU is precisely one whose slope is $(\sqrt{5} - 1)/2$, which is the slope of X_1, the characteristic vector of Q. Can you show that X_1 is also a characteristic vector of P?

We have shown

<u>Theorem 1</u>. The matrix $P = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ maps a vector (x, y) onto a vector PU such that

(1) $$|PU| = \sqrt{5}|U| ;$$

(2) The angle bisector of the angle between the vector U and the vector PU is X_1, a characteristic vector of Q and P. Thus matrix P reflects vector U across vector X_1.

<u>Theorem 2</u>. The vectors U_n and V_n are equally inclined to the vector X_1 whose slope is $(\sqrt{5} - 1)/2$.

<u>Corollary</u>. The vectors V_n are mapped onto vectors $\sqrt{5}\,U_n$ by P and the vectors U_n are mapped onto V_n by P.

7. SOME INTERESTING ANGLES

An interesting theorem is

<u>Theorem 3</u>.

$$\text{Tan}(\text{Tan}^{-1} L_n/L_{n+1} - \text{Tan}^{-1} L_{n+1}/L_{n+2}) = (-1)^n/F_{2n+2}$$

Theorem 4.

$$\mathrm{Tan}(\mathrm{Tan}^{-1} F_n/F_{n+1} - \mathrm{Tan}^{-1} F_{n+1}/F_{n+2}) = (-1)^{n+1}/F_{2n+2}$$

Theorem 5.

$$\mathrm{Tan}^{-1} F_n/F_{n+1} = \sum_{m=1}^{n} (-1)^{m+1} \mathrm{Tan}^{-1} 1/F_{2m}$$

We proceed by mathematical induction. For $n = 1$, it is easy to verify that $\mathrm{Tan}^{-1} 1 = \mathrm{Tan}^{-1} (1/F_2)$.

Assume that Theorem 5 is true for $n = k$; that is, that

$$\mathrm{Tan}^{-1} F_k/F_{k+1} = \sum_{m=1}^{k} (-1)^{m+1} \mathrm{Tan}^{-1} 1/F_{2m}$$

But, by Theorem 4,

$$\mathrm{Tan}^{-1} F_{k+1}/F_{k+2} = \mathrm{Tan}^{-1} F_k/F_{k+1} + \mathrm{Tan}^{-1} (-1)^k/F_{2k+2}$$

Thus, if the induction hypothesis is true, then

$$\mathrm{Tan}^{-1} F_{k+1}/F_{k+2} = \sum_{m=1}^{k} (-1)^{m+1} \mathrm{Tan}^{-1} 1/F_{2m} + \mathrm{Tan}^{-1} (-1)^k/F_{2k+2}$$

$$= \sum_{m=1}^{k+1} (-1)^{m+1} \mathrm{Tan}^{-1} 1/F_{2m}$$

because $\mathrm{Tan}^{-1} (-x) = -\mathrm{Tan}^{-1} x$ and $(-1)^k = (-1)^{k+2}$ and the proof is complete.

8. AN EXTENDED RESULT

Theorem 6. The series

$$A = \sum_{m=1}^{\infty} (-1)^{m+1} \mathrm{Tan}^{-1} 1/F_{2m}$$

converges and $A = \mathrm{Tan}^{-1} (\sqrt{5} - 1)/2$.

Proof: Since the series is an alternating series, and, since $\mathrm{Tan}^{-1} x$ is a continuous increasing function, then

$$\operatorname{Tan}^{-1} 1/F_{2n} > \operatorname{Tan}^{-1} 1/F_{2n+2} \quad \text{and} \quad \operatorname{Tan}^{-1} 0 = 0 .$$

The angle A must lie between the partial sums S_N and S_{N+1} for every $N > 2$ by the error bound in the alternating series, but by Theorem 5, $S_N = \operatorname{Tan}^{-1} F_N/F_{N+1}$. Thus the angles of U_N and U_{N+1} lie on opposite sides of A. By the continuity of $\operatorname{Tan}^{-1} x$, then,

$$\lim_{n\to\infty} \operatorname{Tan}^{-1}(F_n/F_{n+1}) = A = \operatorname{Tan}^{-1}(\sqrt{5} - 1)/2 .$$

Comment: the same result can be obtained simply from

$$\operatorname{Tan}\left[\operatorname{Tan}^{-1} F_n/F_{n+1} - \operatorname{Tan}^{-1}(\sqrt{5} - 1)/2\right] = (-1)^{n+1}[(\sqrt{5} - 1)/2]^{2n+1}$$

Which slope gives a better numerical approximation to $(\sqrt{5} - 1)/2$, F_n/F_{n+1} or L_n/L_{n+1}? Hmmm?

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SOME MORE ELEMENTARY PROBLEMS

B-4 (Proposed by S. L. Basin and Vladimir Ivanoff) Show that

$$\sum_{i=0}^{n} \binom{n}{i} F_i = F_{2n}$$

and generalize.

B-5 (Proposed by L. Moser) Show that, with order taken into account, in getting paid an integral number n dollars, using only one-dollar and two-dollar bills, that the number of different ways is F_{n+1} where F_n is the nth Fibonacci number.

B-9 (Proposed by R. L. Graham) Prove that

$$\sum_{n=2}^{\infty} \frac{1}{F_{n-1}F_{n+1}} = 1 \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}} = 2 .$$

B-10 (Proposed by Stephen Fisk) Prove the "de Moivre-type" identity

$$\left(\frac{L_n + \sqrt{5}\, F_n}{2}\right)^p = \frac{L_{np} + \sqrt{5}\, F_{np}}{2}$$

where L_n denotes the nth Lucas number and F_n denotes the nth Fibonacci number.