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The Quarterly is entered as third class mail at the St. Mary's College Post Office, California, as an official publication of the Fibonacci Association.

EDITORIAL

The Fibonacci Association was formed in order to exchange ideas and stimulate research in Fibonacci numbers and related topics. From the start, the group was active in producing results and it soon became evident that a journal would be highly desirable for the rapid dissemination of this research. Another phase of activity was the gathering of bibliographical material which was extensive. Our present bibliography of over seven hundred items indicates that the Fibonacci numbers have long sustained a wide interest with papers appearing in many languages and contributors ranging from curious amateurs to serious researchers.

We hope that the journal may serve as a focal point for widespread interest in Fibonacci numbers, especially with respect to new results, research proposals and challenging problems. In addition we wish to help nurture beginners in the fundamentals of Fibonacci numbers, using the field of recurrent sequences as a background in which various basic concepts of simple research may be illustrated.

Mathematics teachers and students of all levels are encouraged to share our enthusiasm and to develop an interest in arithmetic number sequences through participation in classroom projects, elementary problem sections, and simple exploration of Fibonacci number facts. The thrill of discovery is wonderful; and devising a good proof is satisfying, even if the discovery or proof is not new, so long as it is an original experience for the student.

Manuscripts submitted for publication should be typewritten, double-spaced and carefully prepared. Authors are encouraged to keep a complete copy of their manuscript. The articles should be written in a style which is more expository than is usually found in mathematical journals. Besides the technical papers there will also be "Problem and Solutions" sections, both elementary and advanced. In addition, there will be research proposals for readers looking for something to investigate. Results therefrom will receive careful attention.

This is a journal for active readers; the editors desire reader participation especially from mathematics teachers and students.

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PART I

A GENERALIZATION OF SEMI-COMPLETENESS FOR INTEGER SEQUENCES

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Notation

In this article, the notation $f_i(\infty)$ will be used to signify

$$\{ f_i \}_1^{\infty} .$$

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Given a sequence $f_i(\infty)$ of positive integers and two auxiliary sequences $k_i(\infty)$ of positive integers and $m_i(\infty)$ of nonnegative integers, we wish to consider the possibility of expanding an arbitrary positive integer n in the form

$$n = \sum_1^M \alpha_i f_i ,$$

where M is finite and each α_i is an integer (zero and negative values allowed) satisfying

$$-m_i \leq \alpha_i \leq k_i \quad \text{for } i = 1, 2, \ldots, M.$$

Throughout the paper, the convention is adopted that $k_i(\infty)$ and $m_i(\infty)$ will always denote given sequences of positive and nonnegative integers, respectively.

As an application of the results to be proved, we shall show that every positive integer n has an expansion in the form

$$n = \sum_1^M \alpha_i F_i^p ,$$

where p is a fixed integer greater than or equal to 2, $F_i(\infty) = \{1, 1, 2, 3, 5, \ldots\}$ is the usual Fibonacci sequence and α_i is an integer satisfying $|\alpha_i| \leq 2^{p-2}$ for each value of i.

DEFINITION 1: A sequence of positive integers $f_i(\infty)$, is said to be quasi-complete with respect to the sequences $k_i(\infty)$ and $m_i(\infty)$ iff (if and only if)

$$0 < n < 1 + \sum_1^N k_i f_i \quad \text{implies}$$

(1) $$n = \sum_1^N \alpha_i f_i \quad \text{with } \alpha_i \text{ integral and}$$

(2) $$-m_i \leq \alpha_i \leq k_i \quad \text{for} \quad i = 1, 2, \ldots, N.$$

The purpose of the present paper is to obtain a characterization of quasi-completeness and to investigate the conditions under which the representation in (1) is unique. Moreover, we will also show that any nondecreasing sequence of positive integers $f_i(\infty)$ which is either complete or semi-complete must also be quasi-complete.

Before proceeding to the proof of the characterization theorem, we recall some pertinent definitions and a lemma.

DEFINITION 2: (Reference 1). A sequence of positive integers $f_i(\infty)$ is complete iff every positive integer n has a representation in the form

(3) $$n = \sum_1^\infty c_i f_i, \quad \text{where each } c_i \text{ is either zero}$$

or one.

DEFINITION 3: (Reference 2). A sequence of positive integers $f_i(\infty)$ is semi-complete with respect to the sequence of positive integers $k_i(\infty)$, iff every positive integer n has a representation

(4) $$n = \sum_{1}^{\infty} c_i f_i \quad , \text{ where each } c_i \text{ is a nonnegative}$$

integer satisfying

(5) $$0 \leq c_i \leq k_i \quad .$$

LEMMA 1: (Alder, Ref. 2, pp. 147-8). Let $f_i(\infty)$ be a given sequence of positive integers with $f_1 = 1$ and such that

$$f_{p+1} \leq 1 + \sum_{1}^{p} k_i f_i \quad \text{for } p = 1, 2, 3, \ldots,$$

where $k_i(\infty)$ is a fixed sequence of positive integers. Then for any positive integer n satisfying the inequality

$$0 < n < 1 + \sum_{1}^{N} k_i f_i \, ,$$

there exist nonnegative integers, $\alpha_i(N)$, such that

$$n = \sum_{1}^{N} \alpha_i f_i \quad \text{and} \quad 0 \leq \alpha_i \leq k_i$$

for i 1, 2, ... , N.

The following theorem gives a necessary and sufficient condition for quasi-completeness.

THEOREM 1: For given sequences, $k_i(\infty)$ and $m_i(\infty)$, the sequence of positive integers $f_i(\infty)$ with $f_1 = 1$ is quasi-complete iff

(6) $$f_{p+1} \leq 1 + \sum_{1}^{p} (k_i + m_i) f_i \quad \text{for } p = 1, 2, 3 \ldots$$

PROOF. Assume condition (6) is satisfied, and let n be a fixed positive integer satisfying

$$0 < n < 1 + \sum_1^N k_i f_i . \quad \text{Then}$$

$$0 < n + \sum_1^N m_i f_i < 1 + \sum_1^N (k_i + m_i) f_i ,$$

and Lemma 1 implies

$$n + \sum_1^N m_i f_i = \sum_1^N \beta_i f_i , \tag{7}$$

where each β_i is a non-negative integer satisfying $0 \leq \beta_i \leq k_i + m_i$ for each $i = 1, 2, \ldots, N$. Thus

$$n = \sum_1^N (\beta_i - m_i) f_i , \text{ with } -m_i \leq \beta_i - m_i \leq k_i , \tag{8}$$

and the identification $\alpha_i = \beta_i - m_i$ for $i = 1, 2, \ldots, N$ shows that $f_i(\infty)$ is quasi-complete.

Conversely, assume $f_i(\infty)$ is quasi-complete. Then by Definition 1, the inequality

$$0 < n < 1 + \sum_1^N k_i f_i \quad \text{implies}$$

$$n = \sum_1^N \alpha_i f_i \quad \text{with } -m_i \leq \alpha_i \leq k_i ,$$

and we wish to show that (6) is satisfied.

For a proof by contradiction, assume (6) does not hold; then, there exists an integer r greater than zero such that

$$f_{r+1} > 1 + \sum_1^r (k_i + m_i) f_i . \tag{9}$$

Hence,

$$0 < f_{r+1} - \sum_1^r m_i f_i - 1 < f_{r+1} < 1 + \sum_1^{r+1} k_i f_i ,$$

and by the quasi-completeness of $f_i(\infty)$,

$$(10) \qquad f_{r+1} - \sum_1^r m_i f_i - 1 = \sum_1^{r+1} \alpha_i f_i \text{ with } -m_i \leq \alpha_i \leq k_i .$$

Now, in the representation (10), $\alpha_{r+1} > 0$, for, if not, then

$$f_{r+1} = \alpha_{r+1} f_{r+1} + 1 + \sum_1^r (\alpha_i + m_i) f_i$$

$$\leq 1 + \sum_1^r (\alpha_i + m_i) f_i \leq 1 + \sum_1^r (k_i + m_i) f_i$$

in violation of assumption (9). Thus, from (10)

$$(11) \qquad - \sum_1^r (\alpha_i + m_i) f_i - 1 = (\alpha_{r+1} - 1) f_{r+1} \geq 0 .$$

But the left hand side of (11) is clearly ≤ -1, giving the desired contradiction. We conclude that (6) must be satisfied for all values of $p \geq 1$.

For nondecreasing sequences, quasi-completeness can be rephrased in terms of semi-completeness according to the following Corollary:

COROLLARY 1: A nondecreasing sequence of positive integers $f_i(\infty)$ is quasi-complete with respect to the sequences

$$k_i(\infty) \text{ and } m_i(\infty)$$

iff $f_i(\infty)$ is semi-complete with respect to the sequence

$$\{k_i + m_i\}$$

PROOF: From [2] and Theorem 1, the necessary and sufficient condition for both statements is inequality (6).

COROLLARY 2: If a nondecreasing sequence of positive integers $f_i(\infty)$ with $f_1 = 1$ is complete, then it is also quasi-complete with respect to arbitrary sequences, $k_i(\infty)$ and $m_i(\infty)$.

PROOF: By Theorem 1 of [1],

$$f_{p+1} \leq 1 + \sum_1^p f_i \leq 1 + \sum_1^p (k_i + m_i) f_i \quad \text{for } p = 1, 2, \ldots$$

for arbitrary sequences $k_i(\infty)$ and $m_i(\infty)$, since $k_i \geq 1$ and $m_i \geq 0$ for all $i \geq 1$.

COROLLARY 3: If a nondecreasing sequence of positive integers $f_i(\infty)$ with $f_1 = 1$ is semi-complete with respect to the sequence of positive integers $k_i(\infty)$, then it is also quasi-complete with respect to the same sequence $k_i(\infty)$ and any sequence $m_i(\infty)$ of nonnegative integers.

PROOF: From Theorem 1 of [2],

$$f_{p+1} \leq 1 + \sum_1^p k_i f_i \leq 1 + \sum_1^p (k_i + m_i) f_i ,$$

and the Corollary is immediate from the characterization of quasi-completeness. Alternatively, Corollary 1 implies the result since semi-completeness with respect to $k_i(\infty)$ implies semi-completeness with respect to $\{k_i + m_i\}$.

Before discussing uniqueness, we note that, for given sequences $k_i(\infty)$ and $m_i(\infty)$, quasi-completeness is a sufficient condition for every positive integer to possess a representation in the form of equation (1). However, if the m_i are not all zero, then quasi-completeness is not necessary for such representations even in the case of a nondecreasing sequence $f_i(\infty)$. For, let $k_i = m_i = 1$ for all $i \geq 1$, and consider the particular sequence

$$f_i(\infty) = \{1, 10, 100, 101, 102, 103, 104, 105, \ldots\}$$

Then the inequality

$$f_{p+1} \leq 1 + 2 \sum_1^p f_i \quad \text{is not satisfied for } p=1, 2..;$$

nevertheless, any positive integer n has a representation in the prescribed form:

$$n = (102-101) + (104-103) + \ldots + [(100+2n)-(100+2n-1)]$$

Clearly, the same situation obtains for any sequence $f_i(\infty)$ which contains all consecutive integers after some fixed index $n = n_0$, where n_0 may be arbitrarily large. This shows that in order to obtain a necessary condition which holds for all members of the sequence, some additional constraint must be introduced. The one chosen in the above theorem requires that whenever

$$n < 1 + \sum_1^N k_i f_i \text{, the representation for n can be}$$

accomplished in terms of the first N members of the sequence. Thus, for a quasi-complete sequence, every positive integer which is

$$\leq \sum_1^N k_i f_i$$

can be represented in the proper form using only the terms $f_1, f_2, f_3, \ldots\ldots f_N$. But the largest number that can be represented in the proper form using only these terms is

$$\sum_1^N k_i f_i$$

so that, in this sense, the condition is the best possible.

In order to discuss uniqueness of the representation, we introduce, for given sequences $k_i(\infty)$ and $m_i(\infty)$, the particular sequence of positive integers $\varphi_i(\infty)$, defined by

$$(12)\qquad \begin{aligned} \varphi_1 &= 1 \\ \varphi_{p+1} &= 1 + \sum_1^p (k_i + m_i)\,\varphi_i, \quad \text{for } p \geq 1. \end{aligned}$$

It is straightforward to show that the terms of this sequence may also be written in the equivalent form:

$$(13)\qquad \begin{aligned} \varphi_1 &= 1 \\ \varphi_{p+1} &= \prod_{i=1}^{p} (1 + k_i + m_i), \quad \text{for } p \geq 1. \end{aligned}$$

DEFINITION 4: For given sequences $k_i(\infty)$ and $m_i(\infty)$, a sequence of positive integers $f_i(\infty)$ will be said to possess the <u>uniqueness property</u> iff for any $N > 0$, the equation

$$(14)\qquad \sum_1^N \alpha_i f_i = \sum_1^N \beta_i f_i, \text{ with } -m_i \leq \alpha_i \leq k_i$$

$$\text{and } -m_i \leq \beta_i \leq k_i, \qquad i = 1, 2, \ldots, N$$

implies $\alpha_i = \beta_i$ for $i = 1, 2, \ldots, N$. (In other words, every integer, positive or negative, which possesses a representation in the required form has only one such representation in that form.)

THEOREM 2: Let $k_i(\infty)$ and $m_i(\infty)$ be given and let $f_i(\infty)$ be a quasi-complete sequence of positive integers with $f_1 = 1$.

Then $f_i(\infty)$ possesses the uniqueness property iff

$$(15)\qquad f_i = \varphi_i \quad \text{for } i = 1, 2, 3, \ldots.$$

PROOF: Assume $f_i(\infty)$ possesses the uniqueness property and that $f_i = \varphi_i$ does not hold for all $i \geq 1$. Then there exists a least integer $N > 0$ such that $F_N \neq \varphi_N$. (Note that $N > 1$). From the quasi-completeness, we have

$$0 < f_N \leq 1 + \sum_1^{N-1} (k_i + m_i) f_i = 1 + \sum_1^{N-1} (k_i + m_i) \varphi_i = \varphi_N$$

Since $f_N \neq \varphi_N$, we must have $f_N < \varphi_N$ and

$$0 < f_N < 1 + \sum_1^{N-1} (k_i + m_i) \varphi_i .$$

By Lemma 1, f_N can be written in the form

(16) $f_N = \sum_1^{N-1} \gamma_i \varphi_i = \sum_1^{N-1} \gamma_i f_i$ where each γ_i is a nonnegative integer satisfying $0 \leq \gamma_i \leq k_i + m_i$.

Hence,

(17) $f_N - \sum_1^{N-1} m_i f_i = \sum_1^{N-1} (\gamma_i - m_i) f_i$, where

$$-m_i \leq \gamma_i - m_i \leq k_i .$$

Applying the uniqueness property to (17), we find that $\gamma_i - m_i = -m_i$ or $\gamma_i = 0$ for $i = 1, 2, \ldots, N-1$. Thus, from (16), $f_N = 0$, a contradiction, and we conclude that $f_i = \varphi_i$ for all $i \geq 1$.

For the converse, we must show that $\varphi_i(\infty)$ possesses the uniqueness property. The proof is by contradiction. If $\varphi_i(\infty)$ does not possess the uniqueness property, then there is a <u>least</u> integer $N > 0$ such that $\alpha_i(N)$ and $\beta_i(N)$ exist having the property

(18) $\sum_1^{N} \alpha_i \varphi_i = \sum_1^{N} \beta_i \varphi_i$ with $-m_i \leq \alpha_i \leq k_i$

and $-m_i \le \beta_i \le k_i$, $(i = 1, 2, \ldots, N)$

and $\sum_1^N |\alpha_i - \beta_i| \neq 0$.

Clearly, $N > 1$, since $\alpha_1 \varphi_1 = \beta_1 \varphi_1$ implies $\alpha_1 = \beta_1$. Moreover, we assert that $\alpha_N \neq \beta_N$ in (18). For if $\alpha_N = \beta_N$, then

$$\sum_1^{N-1} \alpha_i \varphi_i = \sum_1^{N-1} \beta_i \varphi_i, \text{ with } -m_i \le \alpha_i \le k_i,$$

$$-m_i \le \beta_i \le k_i, \quad (i = 1, 2, \ldots, N-1)$$

and $\sum_1^{N-1} |\alpha_i - \beta_i| \neq 0$. But this contradicts our choice of N as the smallest upper limit affording two distinct representations. Hence $\alpha_N \neq \beta_N$.

From (18)

$$(\beta_N - \alpha_N)\varphi_N = \sum_1^{N-1} (\alpha_i - \beta_i)\varphi_i, \tag{19}$$

and therefore,

$$\varphi_N \le |\beta_N - \alpha_N| \varphi_N \le \sum_1^{N-1} |\alpha_i - \beta_i| \varphi_i \tag{20}$$

$$\le \sum_1^{N-1} (k_i + m_i)\varphi_i = \varphi_N - 1,$$

giving a contradiction.

EXAMPLES: (a) As our first example, we consider the sequence of pth powers of the Fibonacci numbers, where $p \ge 2$. It is known [1] that the sequence $F_i(\infty)$, which is defined as $\{1, 1, 2, 3, 5, \ldots\}$, is complete; therefore, every positive integer n has a representation in the form

$$n = \sum_{1}^{M} c_i F_i \text{, where each } c_i \text{ is either 0 or 1.}$$

To generalize this result, we leave it to the reader to verify the following inequality:

$$(21) \qquad F_{n+1}^{p} \leq 1 + 2^{p-1} \sum_{1}^{n} F_i^p \text{, where } p \text{ is a fixed integer}$$

greater than or equal to 1. From (21), it is clear that, for sequences $k_i(\infty)$ and $m_i(\infty)$ defined by $k_i = m_i$,

$$m_i = 2^{p-2}, \quad \text{for all } i \geq 1, \text{ the sequence}$$

$F_i^p(\infty)$ is quasi-complete. Thus every positive integer n has a representation in the form

$$(22) \qquad n = \sum_{1}^{\infty} \alpha_i F_i^p \text{, where } \alpha_i \text{ is an integer}$$

$$\text{satisfying } |\alpha_i| \leq 2^{p-2}, \text{ for } i \geq 1.$$

Moreover, from Theorem 1, if N is chosen so that

$$0 < n < \sum_{1}^{N} 2^{p-2} F_i^p ,$$

then n has at least one representation in the form (22) which uses only the terms

$$F_1^p,\ F_2^p,\ \ldots\ldots\ldots, F_N^p .$$

In particular, $0 < n < 1 + \sum_{1}^{N} F_i^2$ implies

$$n = \sum_{1}^{N} \alpha_i F_i^2 \text{, where each } \alpha_i \text{ is}$$

either -1, 0, or +1.

(b) To illustrate Theorem 2, let $k_i(\infty)$ and $m_i(\infty)$ be defined by $k_i \equiv m_i \equiv 1$ for all $i \geq 1$. Then, if a given sequence $f_i(\infty)$ is quasi-complete with respect to $k_i(\infty)$ and $m_i(\infty)$, every positive integer n has a representation

$$n = \sum_1^\infty \alpha_i f_i, \text{ where each } \alpha_i \text{ is either } -1, 0, \text{ or } +1$$

Next, define [compare (13)]

$$(23) \qquad \begin{aligned} \varphi_1 &= 1 \\ \varphi_{N+1} &= \prod_1^n (1 + k_i + m_i) = \prod_1^n 3 = 3^n, \text{ for } n \geq 1. \end{aligned}$$

Then $\varphi_n = 3^{n-1}$ for all $n \geq 1$, and since $\varphi_{n+1} = 1 + 2\sum_1^n \varphi_i$, the sequence $\varphi_i(\infty)$ is quasi-complete with respect to the unity sequences, $k_i(\infty)$ and $m_i(\infty)$. Moreover, according to Theorem 2, representations are unique in the sense that if

$$\sum_1^M \alpha_i \varphi_i = \sum_1^M \beta_i \varphi_i \quad \text{with } |\alpha_i| \leq 1 \text{ and } |\beta_i| \leq 1 \text{ for}$$

$i \geq 1$, then $\alpha_i = \beta_i$ for $i = 1, 2, \ldots, M$.

Combining Theorems 1 and 2, we have that every integer n satisfying

$$0 < n < 1 + \sum_1^N 3^{i-1} = \frac{3^N + 1}{2}$$

(namely, the integers $1, 2, 3, \ldots, \frac{3^N - 1}{2}$) has a unique representation in the form

$$n = \sum_1^N \alpha_i 3^{i-1} \quad \text{with each } \alpha_i = -1, 0, \text{ or } +1$$

The reader may note that this result provides a solution to Bachet's weighing problem [3] . It is also left to the reader to interpret the quasi-complete sequence $\varphi_i(\infty)$ of (13) as the solution of a dual-pan weighing problem with the constraint that at most k_i weights of magnitude φ_i can be used in the right pan, at most m_i weights of magnitude φ_i can be used in the left pan, and every integral number of pounds less than or equal to

$$\sum_1^N k_i \varphi_i$$

must be weighable using only the weights, $\varphi_1, \varphi_2, \ldots \varphi_N$.

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[2] H.L. Alder, The Number System in More General Scales, Mathematics Magazine, Vol. 35, 1962, pp. 145-151.

[3] G.H. Hardy and E.M. Wright, An Introduction to the Theory of Numbers, Third Edition, Oxford University Press, pp. 115-117.

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HAVE YOU SEEN?

J. Riordan, Generating Functions for Powers of Fibonacci Numbers, Duke Mathematical Journal, Vol. 29, 1962, pp. 5-12. Reprints available from author.

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16 EXPANSION OF ANALYTIC FUNCTIONS IN POLYNOMIALS ASSOCIATED WITH FIBONACCI NUMBERS

Paul F. Byrd

1. Introduction. A problem which has long been of fundamental interest in classical analysis is the expansion of a given function f(x) in a series of the form

$$(1.1) \qquad f(x) = \sum_{n=0}^{\infty} b_n P_n(x)$$

where $\{P_n(x)\}$ is a prescribed sequence of polynomials, and where the coefficients b_n are numbers related to f. In particular, the innumerable investigations on expansions of "arbitrary" functions in orthogonal polynomials have led to many important convergence and summability theorems, and to various interesting results in the theory of approximation. (See, for example, Alexits [1], Szegö [2], Rainville [3], and Jackson [4].) Numerous recent studies have also been made on the expansion of analytic functions employing more general sets of polynomials (e.g., see Whittaker [5], or Boas and Buck [6]). There is thus already in existence a great wealth of theory which may be applied when a particular set of polynomials is introduced to accomplish a certain purpose.

In the present article, we shall apply some available results in order to consider the expansion of analytic functions in a series of a certain set of polynomials which can be associated with the famous numbers of Fibonacci. Our primary objective is to illustrate a simple, general technique that may be used to obtain expansions of a given class of functions in terms involving Fibonacci numbers. Some important broad questions and problems concerning convergence and the representability of our polynomial expansions in general will not be discussed, however.

2. Fibonacci Polynomials. By 'Fibonacci polynomials' we shall mean the sequence of polynomials $\{\varphi_k(x)\}$,

($k = 0, 1, \ldots\ldots$) satisfying the recurrence relation[1]

$$(2.1) \qquad \varphi_{k+2}(x) - 2x\,\varphi_{k+1}(x) - \varphi_k(x) = 0, \quad -\infty < x < \infty$$

with initial conditions

$$(2.2) \qquad \varphi_0(x) = 0, \quad \varphi_1(x) = 1.$$

In the special case when $x = 1/2$, equations (2.1) and (2.2) clearly reduce to the well-known relations [7] that furnish the Fibonacci numbers $0, 1, 1, 2, 3, \ldots.,$ which we shall denote by $\varphi_k(1/2)$ or F_k.

A generating function defining the polynomials $\varphi_k(x)$ is

$$(2.3) \qquad \frac{s}{1 - 2xs - s^2} = \sum_{k=0}^{\infty} \varphi_k(x)\, s^k .$$

Now since the left member of (2.3) changes sign if x is replaced by $(-x)$ and s by $(-s)$, we have

$$(2.4) \qquad \varphi_k(-x) = (-1)^{k+1}\,\varphi_k(x)$$

thereby showing that $\varphi_{k+1}(x)$ is an odd function of x for k odd and an even function of x for k even. Upon expanding the left side of (2.3) and equating coefficients in s, we obtain the explicit formula

$$(2.5) \qquad \varphi_{k+1}(x) = \sum_{m=0}^{[k/2]} \binom{k-m}{m} (2x)^{k-2m}, \quad (k \geq 0),$$

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[1] A related set of polynomials, which satisfies the recurrence relation $y_{k+2}(x) - x\, y_{k+1}(x) - y_k(x) = 0$, was considered in 1883 by Catalan [8]. The name 'Fibonacci polynomials' is also given to solutions of the relation $z_{k+2}(x) = z_{k+1}(x) + x\, z_k(x)$, $z_0(x) = 0$, $z_1(x) = 1$, investigated by Jacobsthal [9].

where $[k/2]$ is the greatest integer $\leq k/2$.

An alternative form for expressing the polynomials $\varphi_k(x)$ may be found by introducing the exponential generating function defined by

$$(2.6) \qquad Y(s,x) = \sum_{k=0}^{\infty} \varphi_k(x) \frac{s^k}{k!}$$

This transforms the recurrence relation (2.1), and the initial conditions (2.2), into the differential equation

$$(2.7) \qquad \frac{d^2Y}{ds^2} - 2x\frac{dY}{ds} - Y = 0$$

with conditions

$$(2.8) \qquad Y(0,x) = 0, \quad \left.\frac{dY}{ds}\right|_{s=0} = 1.$$

The solution of (2.7) thus yields the generating function

$$(2.9) \qquad Y(s,x) = \frac{1}{2\sqrt{(1+x^2)}} [e^{sa_1} - e^{sa_2}],$$

where

$$(2.10) \qquad a_1 = x + \sqrt{(1+x^2)}, \quad a_2 = x - \sqrt{(1+x^2)}.$$

If we now apply the inverse transform

$$(2.11) \qquad \varphi_k(x) = \left.\frac{d^k Y}{ds^k}\right|_{s=0}, \qquad k = 0,1,2,\ldots.$$

$$(2.12) \qquad x = \sinh\omega, \quad \sqrt{(1+x^2)} = \cosh\omega$$

we obtain

$$\varphi_{2k}(x) = \frac{\sinh 2k\omega}{\cosh \omega}$$
$$\text{(2.13)} \qquad\qquad (k = 0, 1, 2, \ldots .)$$
$$\varphi_{2k+1}(x) = \frac{\cosh (2k+1)\omega}{\cosh \omega}$$

3. Some Other Relations. We note, as can easily be shown that the polynomials $\varphi_m(x)$ are related to Chebyshev's polynomials $U_m(x)$ of the second kind[2] [3] by

$$\text{(3.1)} \quad \varphi_0(x) = U_0(ix) = 0, \; \varphi_{m+1}(x) = (-i)^m \, U_{m+1}(ix),$$
$$(i = \sqrt{-1}, \; m \geq 0).$$

The Chebyshev polynomials themselves of course belong to a larger family designated as 'ultraspherical polynomials' or sometimes 'Gegenbauer polynomials' 2] . Unlike those of Chebyshev or of Gegenbauer, however, our Fibonacci polynomials $\varphi_m(x)$ are not orthogonal on any interval of the real axis.

The sequence $\varphi_{k+1}(x)$, $(k = 0, 1, 2, \ldots)$ is a so-called simple set, since the polynomials are of degree precisely k in x, as is seen from (2.5). Thus the linearly independent set contains one polynomial of each degree, and any polynomial $P_n(x)$ of degree n can clearly be expressed linearly in terms of the elements of the basic set; that is, there always exist constants c_k such that the finite sum

$$\text{(3.2)} \qquad P_n(x) = \sum_{k=0}^{n} c_k \, \varphi_{k+1}(x)$$

is a unique representation of $P_n(x)$.

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[2] These polynomials of Chebyshev are not to be confused with the Chebyshev polynomials $T_m(x)$ of the first kind, which are useful in optimal-interval interpolation[10].

Before we seek the explicit expression for the coefficients in the expansion of a given analytic function $f(x)$ in series of our basic set $\{\varphi_{k+1}(x)\}$, it is useful to have x^n in a series of this set. Taking Fibonacci polynomials as defined by formula (2.5), we thus need the easily established reciprocal relation(3),

$$(3.3)\qquad x^n = (1/2^n)\sum_{r=0}^{[n/2]} (-1)^r \binom{n}{r}\frac{n-2r+1}{n-r+1}\varphi_{n+1-2r}(x),\quad n\geq 0,$$

which could also be re-arranged in the form

$$(3.4)\qquad x^n = \sum_{j=0}^{n} \gamma_{nj}\,\varphi_{j+1}(x)$$

that will then contain only even φ's when n is odd, and odd φ's when n is even.

4. Expansion of Analytic Functions. We assume that our arbitrarily given function $f(x)$ can be represented by a power series

$$(4.1)\qquad f(x) = \sum_{n=0}^{\infty} a_n x^n$$

having a radius of convergence of $\zeta \geq 1/2$, with the coefficients a_n expressed by

$$(4.2)\qquad a_n = \frac{f^{(n)}(0)}{n!}\qquad (n = 0, 1, \ldots)$$

Formal substitution of relation (3.3) into (4.1) yields the desired polynomial expansion

$$(4.3)\qquad f(x) \doteq \sum_{k=0}^{\infty} c_k\,\varphi_{k+1}(x),$$

(3) In view of (2.12) and (2.13), this relation is an equivalent form for known expressions for powers of the hyperbolic function sinh ω.

where the coefficients are finally determined from the formula

$$c_k = (k+1) \sum_{j=0}^{\infty} \frac{(-1)^j a_{2j+k}}{2^{2j+k}(j+k+1)} \binom{2j+k}{j} \tag{4.4}$$

Convergence properties of the general basic series (1.1) have been investigated by Whittaker [5], by Boas and Buck [6], and by others. If Whittaker's results are applied to our case, it can be shown that the expansion (4.3) will converge absolutely and uniformly to the function f(x) in $|x| \leq \zeta$ if the series

$$\sum_{n=0}^{\infty} |a_n| V_n(\zeta) \tag{4.5}$$

converges, where $V_n(\zeta)$ is given by

$$V_n(\zeta) = \sum_{j=0}^{\infty} |\gamma_{nj}| M_j(\zeta), \tag{4.6}$$

with

$$M_j(\zeta) = \operatorname*{Max}_{|x|=\zeta} |\varphi_{j+1}(x)|, \tag{4.7}$$

and with γ_{nj} being the coefficients in (3.3) after they have been re-arranged in the form (3.4).

Now, we may also introduce a parameter 2α such that $|2\alpha x| \leq \zeta$, and may thus start with the form

$$f(2\alpha x) = \sum_{n=0}^{\infty} (2^n \alpha^n a_n) x^n = \sum_{n=0}^{\infty} A_n x^n, \tag{4.8}$$

where

$$(4.9) \qquad A_n = \frac{1}{n!} \frac{d^n}{dx^n} f(2\alpha x) \Big|_{x=0}$$

The expansion (4.3) in terms of Fibonacci polynomials then becomes

$$(4.10) \qquad f(2\alpha x) = \sum_{k=0}^{\infty} \beta_k \varphi_{k+1}(x) ,$$

with the coefficients β_k now being determined by the equation

$$(4.11) \quad \beta_k(\alpha) = (k+1) \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^{2j+k}}{j+k+1} a_{2j+k} \binom{2j+k}{j}$$

For our purposes, the form (4.10) is often more convenient than that of (4.3).

If we take $x = 1/2$, the polynomials $\varphi_k(x)$ become the numbers of Fibonacci, $\varphi_k(1/2) = F_k$, so that the series

$$(4.12) \quad f(\alpha) = \sum_{k=0}^{\infty} \beta_k(\alpha) \varphi_{k+1}(1/2) = \sum_{k=0}^{\infty} \beta_k(\alpha) F_{k+1}$$

furnishes a formal expansion of the function $f(\alpha)$ in terms involving Fibonacci numbers. One apparent use of the series expansion (4.12) is for the case in which it is desired to make a given analytic function f serve as a generating function of the Fibonacci-number sequence.

5. Examples. We first consider the function

$$(5.1) \qquad f(x) = e^{2\alpha x} , \qquad (0 < |\alpha| < \infty) ,$$

where

$$(5.2) \qquad a_n = 2^n \alpha^n / n! .$$

The coefficients c_k in (4.4) are then given by the formula

$$(5.3) \qquad c_k = (k+1) \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^{2j+k}}{(2j+k)!\,(j+k+1)} \binom{2j+k}{j}$$

or finally by

$$(5.4) \qquad c_k = \frac{k+1}{\alpha} J_{k+1}(2\alpha), \qquad (k = 0, 1, 2, \ldots\ldots)$$

where J_{k+1} is Bessel's function [11] of order $k+1$. The polynomial expansion (4.3) therefore yields formally

$$(5.5) \quad e^{2\alpha x} = (1/\alpha) \sum_{k=0}^{\infty} (k+1)\, J_{k+1}(2\alpha)\, \varphi_{k+1}(x)$$

$$= (1/\alpha) \sum_{m=1}^{\infty} m\, J_m(2\alpha)\, \varphi_m(x)$$

We note that

$$(5.6) \quad \lim_{m \to \infty} \frac{(m+1)\, J_{m+1}(2\alpha)\, \varphi_{m+1}(x)}{m\, J_m(2\alpha)\, \varphi_m(x)}$$

$$= \lim_{m \to \infty} \frac{(x + \sqrt{1+x^2})}{m}\, \alpha = 0,$$

so that the series (5.5) is convergent for all finite values of x if the parameter α remains also finite.

From (5.5), with the relations

$$(5.7) \quad \begin{aligned} \cosh 2\alpha x &= (e^{2\alpha x} + e^{-2\alpha x})/2, \\ \sinh 2\alpha x &= (e^{2\alpha x} - e^{-2\alpha x})/2, \end{aligned}$$

we immediately obtain the two expansions

$$(5.8) \qquad \cosh 2\alpha x = (1/\alpha) \sum_{m=1}^{\infty} (2m-1)\, J_{2m-1}(2\alpha)\, \varphi_{2m-1}(x)$$

and

$$(5.9) \qquad \sinh 2\alpha x = (1/\alpha) \sum_{m=1}^{\infty} 2m\, J_{2m}(2\alpha)\, \varphi_{2m}(x)$$

Similarly, we have

$$(5.10) \qquad \cos 2\alpha x = (1/\alpha) \sum_{m=1}^{\infty} (-1)^{m+1} (2m-1)\, I_{2m-1}(2\alpha)\, \varphi_{2m-1}(x)$$

and

$$(5.11) \qquad \sin 2\alpha x = (1/\alpha) \sum_{m=1}^{\infty} (-1)^{m+1} (2m)\, I_{2m}(2\alpha)\, \varphi_{2m}(x),$$

where I_k is the modified Bessel function [11] of the first kind of order k.

To produce expansions involving the Fibonacci numbers F_k. we simply set $x = 1/2$. Hence from (5.5), (5.8), (5.9), (5.10) and (5.11), it is seen for $0 < |\alpha| < \infty$ that

$$(5.12) \qquad \begin{aligned} e^{\alpha} &= (1/\alpha) \sum_{m=1}^{\infty} m\, J_m(2\alpha)\, F_m \\ \cosh \alpha &= (1/\alpha) \sum_{m=1}^{\infty} (2m-1)\, J_{2m-1}(2\alpha)\, F_{2m-1} \\ \sinh \alpha &= (1/\alpha) \sum_{m=1}^{\infty} (2m)\, J_{2m}(2\alpha)\, F_{2m} \end{aligned}$$

(5.12)

$$\cos \alpha = (1/\alpha) \sum_{m=1}^{\infty} (-1)^{m+1} (2m-1)\, I_{2m-1}(2\alpha)\, F_{2m-1}$$

$$\sin \alpha = (1/\alpha) \sum_{m=1}^{\infty} (-1)^{m+1} (2m)\, I_{2m}(2\alpha)\, F_{2m}.$$

As $\alpha \longrightarrow 0$, the right-hand sides of (5.12) all become indeterminate forms, but the correct result is obtained in the limit. The particular series expansions (5.5), (5.8), (5.9), (5.10), (5.11) and (5.12) are apparently not found in the literature in the specific form we have presented for our purposes; they could be related, however, to some expansions due to Gegenbauer [11, page 369].

Many higher transcendental functions can also be explicitly developed along similar lines. For instance, without difficulty we may derive the series expansions

$$I_1(\alpha) = (2/\alpha) \sum_{m=1}^{\infty} m\, J_m^2(\alpha)\, F_{2m}$$

(5.13)
$$J_1(\alpha) = (2/\alpha) \sum_{m=1}^{\infty} (-1)^{m+1}\, m\, I_m^2(\alpha)\, F_{2m}$$

for the Bessel functions I_1 and J_1.

The coefficients in the above examples all involve Bessel's functions, but this indeed would not be the case in general. For instance, for

$$|2\alpha x| < 1$$

we can show from (4.10) and (4.11) that

(5.14)
$$\ln (1 + 2\alpha x) = -[r^2/2 + \ln r/\alpha]\, \varphi_1(x) + \sum_{k=1}^{\infty} (-1)^{k+1} r^k [1/k + r^2/(k+2)]\, \varphi_{k+1}(x),$$

where

$$(5.15) \qquad r = \frac{\sqrt{(1 + 4\alpha^2)} - 1}{2\alpha} .$$

With $x = 1/2$, we then have, for $|\alpha| < 1$,

$$(5.16) \qquad \ln(1+\alpha)\,(r/\alpha) = -(r^2/2)\,F_1$$

$$+ \sum_{k=1}^{\infty} (-1)^{k+1} r^k [\,1/k + r^2/(k+2)\,]\,F_{k+1}.$$

6. Another Approach.[4] The coefficients β_k in our basic series expansion (4.10) or (4.12) may be obtained by an alternative procedure which is based on relations (3.1) and certain known properties of the orthogonal polynomials $U_k(x)$. (A good reference giving many properties of U_k is [12]).

If our prescribed function $f(2\alpha x)$ can be expanded in the formal series

$$(6.1) \quad f(2\alpha x) = \sum_{k=0}^{\infty} b_k U_{k+1}(x), \quad |x| < 1, \quad |2\alpha x| \leq \zeta$$

the coefficients b_k are given by

$$(6.2) \quad b_k = (2/\pi)\int_{-1}^{1} f(2\alpha x)\sqrt{(1-x^2)}\,U_{k+1}(x)\,dx, \quad (k = 0, 1 \ldots).$$

With the relations

$$(6.3) \qquad x = \text{cox } v, \quad U_k(x) = (\sin kv)/\sqrt{(1-x^2)}$$

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[4] We could also apply the tools employed in [6] but have written this paper without assuming knowledge of complex-variable methods.

the expression (6.2) becomes

(6.4)
$$b_k(\alpha) = \frac{1}{\pi}\int_0^{\pi} f(2\alpha \cos v)\ [\cos kv - \cos(k+2)v]\ dv.$$

In view of relations (3.1, (4.10) and (6.1), we then have formally,

$$(6.5)\quad \beta_k(\alpha) = \frac{i^k}{\pi}\int_0^{\pi} f(-2\alpha\, i\cos v)\ [\cos v - \cos(k+2)v]\ dv$$

as an equation for β_k in integral form.

In the special case when

$$(6.6)\qquad f(2\alpha x) = e^{2\alpha x}$$

we find

$$(6.7)\quad \beta_k(\alpha) = \frac{i^k}{\pi}\int_0^{\pi} e^{-2\alpha\, i\cos v}[\cos kv - \cos(k+2)v]\ dv$$

$$= J_k(2\alpha) + J_{k+2}(2\alpha) = \frac{k+1}{\alpha}\, J_{k+1}(2\alpha),$$

which is the same result obtained in example (5.5). Usually, however, the integrals (6.5) involving a given function f are not available, so that the expression (4.11) is more often the better procedure for determining the coefficients β_k.

The particular expansions (5.12) and (5.13), or (4.12) in general, turn out to have little use as a means of obtaining efficient approximations for computational purposes. Independent of numerical or physical applications, however, the introduction of Fibonacci numbers into various expressions involving classical functions has a certain interest and fascination in itself.

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PROBLEM DEPARTMENT

P-1. Verify that the polynomials $\varphi_{k+1}(x)$ satisfy the differential equation

$$(1+x^3)\,y'' + 3xy' - k(k+2)y = 0 \qquad (k = 0, 1, 2, \ldots)$$

P-2. Derive the series expansion

$$J_0(x) = \sum_{k=0}^{\infty} (-1)^k \left[I_k^2(\alpha) - I_{k+1}^2(\alpha) \right] F_{2k+1},$$

where J_0 and I_k are Bessel Functions.

P-3. Verify the reciprocal relation

$$x^n = (1/2^n) \sum_{r=0}^{[n/2]} (-1)^r \binom{n}{r} \frac{n-2r+1}{n-r+1} \varphi_{n+1-2r}(x), \quad n \geq 0.$$

P-4. Show that the determinant

$$\varphi_{k+1}(x) = \begin{vmatrix} 2x & i & 0 \ldots\ldots 0 & 0 \\ i & 2x & i \ldots\ldots 0 & 0 \\ 0 & i & 2x \ldots\ldots 0 & 0 \\ \cdot & \cdot & \cdots\cdots & \cdot \\ 0 & 0 & 0 \ldots\ldots 2x & i \\ 0 & 0 & 0 \ldots\ldots i & 2x \end{vmatrix} \qquad k \geq 1,$$

with $\varphi_0(x) = 0$, $\varphi_1(x) = 1$, and where $i = \sqrt{-1}$, satisfies the recurrence relation for $\varphi_k(x)$. Whence derive the expression

$$F_{k+1} = \varphi_{k+1}(1/2) = \begin{vmatrix} 1 & i & 0 \ldots 0 & 0 \\ i & 1 & i \ldots 0 & 0 \\ 0 & i & 1 \ldots 0 & 0 \\ \cdot & \cdot & \cdots\cdots & \cdot \\ 0 & 0 & 0 \ldots 1 & i \\ 0 & 0 & 0 \ldots i & 1 \end{vmatrix} \qquad k \geq 1$$

for the Fibonacci numbers.

P-5. Show that the Fibonacci polynomials may also be expressed by

$$\varphi_{k+1}(x) = \frac{2^k (k+1)!}{\sqrt{1+x^2}\,(2k+1)!} \frac{d^k}{dx^k} (1+x^2)^{k+1/2}, \quad (k \geq 0).$$

OPERATIONAL RECURRENCES INVOLVING FIBONACCI NUMBERS

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The Fibonacci numbers may be defined by the linear recurrence relation

$$F_{n+1} = F_n + F_{n-1} \tag{1}$$

together with the initial values $F_0 = 0$, $F_1 = 1$.

There are some unorthodox ways of making up sequences which involve Fibonacci numbers, and we should like to mention a few of these. For want of a better name, we shall call the recurrences below 'operational recurrences.'

Instead of taking the next term in a sequence as the sum of the two preceding terms, let us suppose the terms of a sequence are functions of x, and define

$$u_{n+1}(x) = D_x (u_n u_{n-1}), \tag{2}$$

where $D_x = d/dx$. As an example, take $u_0 = 1$, $u_1 = e^x$. Then we find

$$u_2 = D(e^x) = e^x$$

$$u_3 = D(e^{2x}) = 2e^{2x}$$

$$u_4 = D(2e^{3x}) = (2)(3)e^{3x}$$

$$u_5 = (1)(1)(2)(3)(5)e^{5x}$$

and we can easily show by induction that

(3) $$u_n = (F_1 F_2 F_3 \cdots\cdots F_n)\, e^{F_n x}, \qquad (u_0 = 1,\ u_1 = e^x).$$

Of course, the addition of exponents led to the appearance of the Fibonacci numbers in this case.

Another operation which we may use is differentiation followed by multiplication with x. We define

(4) $$u_{n+1} = (xD_x)\,(u_n\ u_{n-1}).$$

For an interesting example, let us take

$$u_0 \equiv x^{F_0} = 1, \quad u_1 = x^{F_1} = x\,.$$ Then we claim that

(5) $$u_n = x^{F_n} \prod_{k=1}^{n} F_k^{F_{n+1-k}}, \qquad n \geq 1.$$

Taking x = 1 we obtain the following table of values as a sample:

n	$u_n(1)$
1	$1^1 = 1$
2	$1^1\,1^1 = 1$
3	$1^2\,1^1\,2^1 = 2$
4	$1^3\,1^2\,2^1\,3^1 = 6$
5	$1^5\,1^3\,2^2\,3^1\,5^1 = 60$
6	$1^8\,1^5\,2^3\,3^2\,5^1\,8^1 = 2880$
7	$1^{13}\,1^8\,2^5\,3^3\,5^2\,8^1\,13^1 = 2{,}246{,}400$

For the sake of completeness we give the inductive proof of formula (5). Suppose that

$$u_{n-1} = x^{F_{n-1}} \prod_{k=1}^{n-1} F_k^{F_{n-k}}$$

Then

$$u_n = xD\,(u_{n-1}\,u_{n-2})$$

$$= xD\,\{\, x^{F_{n-1}} x^{F_{n-2}} \prod_{k=1}^{n-1} F_k^{F_{n-k}} \prod_{j=1}^{n-2} F_j^{F_{n-1-j}} \}$$

$$= xD\,\{\, x^{F_n} \}\, F_{n-1}^{F_1} \prod_{k=1}^{n-2} F_k^{F_{n-k}+F_{n-1-k}}$$

$$= F_n\, x^{F_n}\, F_{n-1}^{F_1} \prod_{k=1}^{n-2} F_k^{F_{n+1-k}}$$

$$= F_n^{F_1}\, x^{F_n}\, F_{n-1}^{F_2} \prod_{k=1}^{n-2} F_k^{F_{n+1-k}}$$

$$= x^{F_n} \prod_{k=1}^{n} F_k^{F_{n+1-k}}$$

The only 'tricky' part is to recall that $1 = F_1$ and $F_1 = F_2$ so that the factors may be put together at the last step in the desired form.

Suppose that the function $u_n(x)$ has a power series representation of the form

$$u_n(x) = \sum_{k=0}^{\infty} a_k(n)\, x^k \quad . \tag{6}$$

Imposing the operational recurrence (4) we find readily that the coefficients in (6) must obey the convolution recurrence

$$(7) \qquad a_k(n) = k \sum_{j=0}^{k} a_j(n-2)\, a_{k-j}(n-1) \;.$$

Conversely, if (7) holds then $u_n(x)$ satisfies (4).

As a slight variation of (4) let us next define

$$(8) \qquad u_{n+1} = x^2 D_x (u_n u_{n-1}) \;,$$

and take $u_0 = 1$, $u_1 = x$. Then it is easily shown by induction that

$$(9) \qquad u_n = x^{F_{n+2}-1} \prod_{k=4}^{n+2} (F_k - 2)^{F_{n+3}-k} \;, \quad \text{for } n \geq 2.$$

The reader may find it interesting to derive a formula for the sequence defined by $u_n = u_n(x)$ with

$$(10) \qquad u_{n+1} = x^p D_x(u_n u_{n-1}), \quad u_0 = 1, \; u_1 = x, \; p = 3, 4, 5 \ldots$$

As a final example, let us define a sequence by (4) with $u_0 = 1$, $u_1 = e^x$.

Then the first few values of the function sequence are:

$$u_2 = x\, e^x \;,$$

$$u_3 = (2x^2 + x)\, e^{2x}$$

$$u_4 = (6x^4 + 9x^3 + 2x^2)\, e^{3x}$$

$$u_5 = (60x^7 + 192\, x^6 + 185 x^5 + 62 x^4 + 6x^3)\, e^{5x}$$

and it is evident that $u_n(x)$ equals $P(x)\, e^{F_n x}$, where $P(x)$ is a polynomial of degree $F_{n+1}-1$ in x.

ADDITIONAL FACTORS OF THE FIBONACCI AND LUCAS SERIES

Brother U. Alfred, F.S.C.

In his volume entitled, "Recurring Sequences," D. Jarden (1) has listed known factors of the first 385 Fibonacci and Lucas numbers. The present article has for purpose to explore these numbers for additional factors in the range $p < 3000$.

Initially recourse was had to the results of D. D. Wall (2). His table lists all primes less than 2,000 which have a period other than the maximum. A comparison of these results with Jarden's factorizations indicated that the following additional factors are now known.

FACTOR	NUMBER	FACTOR	NUMBER
1279	L(213)	1823	L(304)
1523	L(254)	1871	L(187)
1553	F(259)	1877	F(313)
1579	L(263)	1913	F(319)
1699	L(283)	1973	F(329)
1733	F(289)	1999	L(333)

The altered factorizations are given below, the newly introduced factors being starred while the remaining residual factors are underlined.

$$L(213) = (2^2)\,(1279^*)\,(688846502588399)\,(\underline{9275009853953658917255851 9})$$

$$L(254) = (3)\,(1523)^*\,(\underline{26487408254541486132499495083633739300801481688547})$$

$$F(259) = (13)\,(73)(149)\,(1553*)(2221)\,(\underline{1230669188181354229694664202889707409030657})$$

$$L(263) = (1579)*\,(\underline{58259567431970886012123727669192696712910749985451 01})$$

L(283) = (1699*) (819046977269431264944632304401
43683491348547590190211271)

F(289) = (577)(1597)(1733*) (6993003378638095531165091
46296699696041517688627857)

L(304) = (607)(1823*)(2207) (1394649074942606274369752
591985187160682041802787493441)

L(187) = (199)(1871*)(3571)(9056742344085065262650973
90431)

F(313) = (1877*)(61685362812877205040156603432943577
707491529123044875479090829)

F(319) = (89)(1913*)(514229)(2373070801850309840641893
5684191808195096087137462113977)

F(329) = (13)(1973*)(2971215073) (33530815263744997367
320800108983382852228390465658077)

L(333) = (2^2) (19) (1999*)(4441)(146521)(1121101)(54018521)
(654168669603048078197865815767570296106159)

The next step was to explore the periods of all primes in the range $2000 < p < 3000$. In carrying out this work the following points were kept in mind:

(1) For primes of the form $10x \pm 1$, the period k(p) of the Fibonacci series is a factor of $p-1$; for primes of the form $10x \pm 3$, k (p) is a factor of $2p + 2$.

(2) For primes of the form $10x \pm 1$, the period is even; for primes of the form $10x \pm 3$, the period has the same power of 2 as is found in $2p + 2$.

The first zero of the Fibonacci series for a prime p is indicated by Z(F,p) , while the first zero for the Lucas series is denoted $Z(L, p)$.

All cases are covered by the following:

(1) If $k(p)$ is of the form $2(2y + 1)$, then $Z(f, p) = k(p)/4$ and $Z(L, p) = k(p)/2$.

(2) If $k(p)$ is of the form $2^2(2y + 1)$, then $Z(F, p) = k(p)/4$ and p is not a factor of the Lucas series.

(3) If $k(p)$ is of the form $2^m(2y + 1)$, $m \geq 3$, then $Z(F, p)$ is $k(p)/2$ and $Z(L, p)$ is $k(p)/4$.

It is to be noted that in the first case $Z(L, p)$ is odd and that $Z(F, p)$ is likewise odd in the second case. In the third case $Z(L, p)$ is even. Knowledge of the first zeros in the Fibonacci and Lucas series leads to a direct conclusion regarding the period of the Fibonacci series.

In the table that follows, the period of the Fibonacci series as well as the first zeros in the Fibonacci and Lucas series are given for all primes in the range $2000 < p < 3000$. All Fibonacci numbers with index a multiple of $Z(F, p)$ have the given prime as a divisor; all Lucas numbers with index an odd multiple of $Z(L, p)$ have the given prime as a divisor.

TABLE OF PERIODS AND ZEROS OF THE FIBONACCI AND LUCAS SERIES IN THE RANGE 2000 to 3000

p	k(p)	Z(F, p)	Z(L, p)
2003	4008	2004	1002
2011	2010	2010	1005
2017	4036	1009	----
2027	1352	676	338
2029	1014	1014	507

TABLE OF PERIODS AND ZEROS

p	k(p)	Z(F,p)	Z(L,p)
2039	2038	2038	1019
2053	4108	1027	----
2063	4128	2064	1032
2069	1034	1034	517
2081	130	130	65
2083	4168	2084	1042
2087	4176	2088	1044
2089	1044	261	----
2099	2098	2098	1049
2111	2110	2110	1055
2113	4228	1057	----
2129	2128	1064	532
2131	2130	2130	1065
2137	4276	1069	----
2141	2140	535	----
2143	4288	2144	1072
2153	4308	1077	----
2161	80	40	20
2179	198	198	99
2203	4408	2204	1102
2207	64	32	16
2213	4428	1107	----
2221	148	37	----
2237	1492	373	----
2239	746	746	373
2243	4488	2244	1122
2251	750	750	375
2267	1512	756	378
2269	324	81	----
2273	4548	1137	----

TABLE OF PERIODS AND ZEROS

p	k(p)	Z(F, p)	Z(L, p)
2281	760	380	190
2287	4576	2288	1144
2293	4588	1147	----
2297	4596	1149	----
2309	2308	577	----
2311	2310	2310	1155
2333	1556	389	----
2339	2338	2338	1169
2341	2340	585	----
2347	4696	2348	1174
2351	2350	2350	1175
2357	4716	1179	----
2371	790	790	395
2377	4756	1189	----
2381	2380	595	----
2383	4768	2384	1192
2389	398	398	199
2393	4788	1197	----
2399	2398	2398	1199
2411	2410	2410	1205
2417	124	31	----
2423	4848	2424	1212
2437	4876	1219	----
2441	1220	305	----
2447	1632	816	408
2459	2458	2458	1229
2467	4936	2468	1234
2473	4948	1237	----
2477	4956	1239	----
2503	5008	2504	1252

TABLE OF PERIODS AND ZEROS

p	k(p)	Z(F, p)	Z(L, p)
2521	120	60	30
2531	2530	2530	1265
2539	2538	2538	1269
2543	5088	2544	1272
2549	2548	637	----
2551	2550	2550	1275
2557	5116	1279	----
2579	2578	2578	1289
2591	518	518	259
2593	5188	1297	----
2609	2608	1304	652
2617	5236	1309	----
2621	1310	1310	655
2633	5268	1317	----
2647	5296	2648	1324
2657	5316	1329	----
2659	886	886	443
2663	1776	888	444
2671	2670	2670	1335
2677	5356	1339	----
2683	5368	2684	1342
2687	1792	896	448
2689	896	448	224
2693	5388	1347	----
2699	2698	2698	1349
2707	5416	2708	1354
2711	2710	2710	1355
2713	5428	1357	----
2719	2718	2718	1359
2729	682	682	341

TABLE OF PERIODS AND ZEROS

p	k(p)	Z(F,p)	Z(L,p)
2731	390	390	195
2741	2740	685	----
2749	916	229	----
2753	1836	459	----
2767	5536	2768	1384
2777	1852	463	----
2789	164	41	----
2791	2790	2790	1395
2797	5596	1399	----
2801	1400	700	350
2803	5608	2804	1402
2819	2818	2818	1409
2833	5668	1417	----
2837	5676	1419	----
2843	5688	2844	1422
2851	2850	2850	1425
2857	5716	1429	----
2861	1430	1430	715
2879	2878	2878	1439
2887	5776	2888	1444
2897	5796	1449	----
2903	5808	2904	1452
2909	2908	727	----
2917	5936	1459	----
2927	5856	2928	1464
2939	2938	2938	1469
2953	5908	1477	----
2957	5916	1479	----
2963	5928	2964	1482
2969	424	212	106
2971	2970	2970	1485
2999	2998	2998	1499

The revised factorizations of Fibonacci numbers resulting from the information in the preceding table are listed below.

F(229) = (457)(2749*)(<u>256799205151071273644115114294</u>
<u>714688654853</u>)

F(261) = (2) (17)(173)(2089*)(514229)(3821263937)
(<u>65082172574960442149015615136409</u>)

F(305) = (5) (2441*)(4513)(555003497) (<u>806206763478084</u>
<u>21095221688408565244445343042761</u>)

F(373) = (2237*)(<u>17915908137997202476959938229552750</u>
<u>8296287028193832492595657294050015005389</u>)

The revised factorizations of Lucas series numbers resulting from the present investigation are given herewith.

L(190) = (3) (41) (2281*)(29134601)(<u>62403963764184557472</u>
<u>8492521</u>)

L(199) = (2389*) (<u>16230214517045729046276217142808933</u>
<u>1241</u>)

L(224) = (1087)(2689*)(4481)(<u>4966336310413757728406317</u>
<u>515606275329</u>)

L(259) = (29) (2591*)(54018521)(<u>33066690054689811646</u>
<u>0968438563218940272271</u>)

L(338) = (3) (2027*)(90481)(<u>7893870715125946824266428</u>
<u>3515154949332581380084893707250688723</u>)

L(341) = (199)(2729*) (3010349) (<u>1125412839062525454792681</u>
<u>92813140395290253805955295249179369</u>)

L(350) = (3) (41) (281) (401) (2801*)(57061)(12317523121)
(5125653689671712991097651838516766450351
8615201)

L(373) = (2239*)(400254035815230315691149177295200668
273516094100108818109883490482374673843 9)

L(378) = (2) (3^4)(83)(107)(281)(1427)(2267*)(11128427)
(33541154206156 83)(610771532623976049480644 6
7593047434562952 3)

CONCLUSION

The present work has continued the table of Wall in systematically determining the periods of primes beyond 2000 and less than 3000. In addition, information has been provided regarding the first zeros of the Fibonacci and Luc as series in this same region.

As a result of Wall's data and its extension in this paper, additional factorizations have been found for Fibonacci and Lucas numbers as given in Jarden's tables. In particular L (263), F(313), F(373), L(199), and L(373) which were previously unfactored have now been shown to be composite.

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(1) Dov Jarden: Recurring Sequences. Riveon Lematematika 1958.

(2) D. D. Wall: Fibonacci Series Modulo m. The American Mathematical Monthly, Vol. 67, No. 6, June-July 1960, pp. 525-532.

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HAVE YOU SEEN?

S.K. Stein, The Intersection of Fibonacci Sequences, Michigan Math. Journal, 9 (1962), Dec., No. 4, 339–402.

L. Carlitz, Generating Functions for Powers of Fibonacci Numbers, Duke Math. Journal Vol. 29 (1962), Mar., No. 1, pp. 5–12.

ON THE FORM OF PRIMITIVE FACTORS OF FIBONACCI NUMBERS

Brother U. Alfred

The factorization of members of the Fibonacci series : 1, 1, 2, 3, 5, 8, ... can be greatly facilitated if the form of the factors is known. Given a Fibonacci number F_n, if n is composite a first step may be taken by dividing out the value of all Fibonacci numbers F_m for which $m \mid n$. There remains a quotient whose factors are primes that have not yet appeared in the Fibonacci series. Following Jarden (Recurring Sequences, p. 8), let us call these primitive prime divisors.

If k(p) is the period for any such primitive prime divisor of F_n, it would follow that:

$$n \mid k(p)$$

If p is of the form $10x \pm 1$, $k(p) \mid p-1$, so that $n \mid p-1$. If p is of the form $10x \pm 3$, $k(p) \mid 2(p+1)$ so that $n \mid 2(p+1)$. Three cases will be distinguished: (A) n an odd quantity; (B) n of the form $2(2r+1)$; (C) n of the form

$$2^m(2r+1), \quad m \geq 2.$$

(A) <u>n odd</u>

For p of the form $10x \pm 1$, $p-1 = nk$ or $p = nk+1$. Since, however, n is odd, k would have to be even to give a prime, so that p would have to be of the form:

$$p = 2nk + 1, \quad (k = 1, 2, 3, \ldots\ldots)$$

If p is of the form $10x \pm 3$, $2(p+1) = kn$. Since n is odd, k is even. Letting $k = 2k'$,

$$p = k'n - 1$$

But again, k' will have to be even if p is to be odd. Thus combining the results for both cases, when n is odd the primitive factors of F_n are of the form:

$$p = 2kn \pm 1, \quad (k = 1, 2, 3, \ldots\ldots)$$

(B) $n = 2(2r+1)$

For p of the form $10x \pm 1$, $p-1 = 2k(2r+1)$

or $p = kn+1$, $(k = 1, 2, 3, \ldots\ldots)$

For $p = 10x \pm 3$, $2(p+1) = 2k(2r+1)$

or $p = k(2r+1) - 1$

To have a prime, k must be even, so that

$$p = nk-1, \quad (k = 1, 2, 3, \ldots\ldots)$$

Hence in case $n = 2(2r+1)$,

$$p = nk \pm 1, \quad (k = 1, 2, 3, \ldots\ldots)$$

(C) $n = 2^m(2r+1)$, $m \geq 2$.

For p of the form $10x \pm 1$, $p-1 = 2^m k\,(2r+1)$

or $p = nk+1$, $(k = 1, 2, 3, \ldots\ldots)$

For p of the form $10x \pm 3$, $2(p+1) = 2^m k(2r+1)$

or $p = 2^{m-1}k(2r+1) - 1$

so that $p = \dfrac{nk}{2} - 1$, $(k = 1, 2, 3, \ldots\ldots)$

VERIFICATION

These forms were verified for all primitive prime factors found in Jarden's Tables (Recurring Sequences) up to $n = 100$. A sampling of the results is indicated in the following table where the notation at the right of the primitive prime factor gives the value of k and the sign of the form used.

n	FORM OF FACTORS	FACTORS
40	$40k \pm 1$, $20k - 1$	2161 (54, -)
41	$82k \mp 1$	2789 (34, +), 59369(724, +)
42	$42k \mp 1$	211(5, +)

n	FORM OF FACTORS	FACTORS
43	$86k \pm 1$	433494437 (5040633, -)
44	$44k \pm 1$, $22k-1$	43(2, -), 307(14, -)
45	$90k \pm 1$	109441 (1216, +)
46	$46k \pm 1$	139(3, +), 461 (10, +)
47	$94k \pm 1$	2971215073 (31608671, -)
48	$48k \pm 1$, $24k-1$	1103(46, -)
49	$98k \pm 1$	97(1, -), 6168709 (62946, +)
50	$50k \pm 1$	101(2, +), 151 (3, +)

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FIBONACCI CENTURY MARK REACHED

It is with a sense of satisfaction that the editors of the Fibonacci Quarterly announce the complete factorization of the first hundred Fibonacci numbers. For the most part, this data is found in the volume, "Recurring Sequences", of D. Jarden. The finishing touches have been provided by John Brillhart of the University of San Francisco whose work was done on the computer at the University of California. This is just a small portion of his extensive factorizations of Fibonacci numbers, but it is a welcome contribution especially at this juncture.

The results are as follows: (* indicates a new factor)

F_{71} =(6673*)(46165371073*) (error in table)

F_{79} = (157)(92180471494753) (final factor a prime)

F_{83} = 99194853094755497 (a prime)

F_{89} = (1069)(1665088321800481) (final factor a prime)

F_{91} = (13^2) (233)(741469*) (159607993*)

F_{93} = (2) (557)(2417)(4531100550901)(final factor a prime)

F_{95} = (5) (37)(113)(761)(29641*)(67735001*) (error in table)

F_{97} = (193)(389)(3084989*)(361040209*)

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Edited by Verner E. Hoggatt, Jr.
San Jose State College

Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, Calif. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-1. Proposed by H. W. Gould, West Virginia University, Morgantown, W. Va.

Find a formula for the nth non-Fibonacci number, that is, for the sequence 4, 6, 7, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 22, 23, ...
(See paper by L. Moser and J. Lambek, American Mathematical Monthly, vol. 61 (1954), pp. 454-458).

H-2. Proposed by L. Moser and L. Carlitz, University of Alberta, Edmonton, Alberta, and Duke University, Durham, N.C. (See also C.S. Ogilvy: Tomorrow's Mathematics, p. 100).

Resolve the conjecture: There are no Fibonacci numbers which are integral squares except 0, 1, and 144.

H-3. Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Show $F_{2n-2} < F_n^2 < F_{2n-1}$, $\quad n \geq 3$;

$$F_{2n-1} < L_{n-1}^2 < F_{2n}, \qquad n \geq 4,$$

where F_n and L_n are the nth Fibonacci and nth Lucas numbers, respectively.

H-4. Proposed by I.D. Ruggles, San Jose State College, San Jose, Calif.

Prove the identity:

$$F_{r+1}F_{s+1}F_{t+1} + F_rF_sF_t - F_{r-1}F_{s-1}F_{t-1} = F_{r+s+t}$$

Are there any restrictions on the integral subscripts?

H-5. Proposed by Terry Brennan, Lockheed Missiles Space Co., Sunnyvale, Calif.

(i) If $[{}_mF_n] = \dfrac{(F_mF_{m-1}\cdots F_1)}{(F_nF_{n-1}\cdots F_1)(F_{m-n}F_{m-n-1}\cdots F_1)}$

then $2\,[{}_mF_n] = [{}_{m-1}F_n]\,L_n + [{}_{m-1}F_{n-1}]\,L_{m-n}$,

where F_n and L_n are the nth Fibonacci and the nth Lucas numbers, respectively.

(ii) Show that this generalized binomial coefficient $[{}_mF_n]$ is always an integer.

H-6. Proposed by Brother U. Alfred, St. Mary's College, Calif.

Determine the last three digits, in base seven, of the millionth Fibonacci number.

H-7. Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

If F_n is the nth Fibonacci number find

$$\lim_{n \to \infty} \sqrt[n]{F_n} = L$$

and show that

$$\sqrt[2n]{\sqrt{5}\, F_{2n}} < L < \sqrt[2n+1]{\sqrt{5}\, F_{2n+1}} \qquad \text{for } n \geq 2.$$

H-8. Proposed by Brother U. Alfred, St. Mary's College, Calif.

Prove

$$\begin{vmatrix} F_n^2 & F_{n+1}^2 & F_{n+2}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \\ F_{n+2}^2 & F_{n+3}^2 & F_{n+4}^2 \end{vmatrix} = 2(-1)^{n+1},$$

where F_n is the nth Fibonacci number.

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FIBONACCI ARTICLES SOON TO APPEAR

A. F. Horadam, Complex Fibonacci Numbers and Fibonacci Quaternions, The American Mathematical Monthly.

S. L. Basin, The Appearance of Fibonacci Numbers and the Q-Matrix in Electrical Network Theory, Mathematics Magazine, March, 1963.

S. L. Basin, An Application of Continuants as a Link between Chebyshev and Fibonacci, Mathematics Magazine

S. L. Basin, Generalized Fibonacci Numbers and Squared Rectangles, American Mathematical Monthly.

D. Zeitlin, On Identities for Fibonacci Numbers. Classroom Notes, American Mathematical Monthly.

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PART II

Edited by Dmitri Thoro
San Jose State College

DIVISIBILITY I

Many of the most interesting properties of the famous Fibonacci numbers ($F_1 = F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $\ldots$, $F_{i+1} = F_i + F_{i-1}$, $\ldots$.) depend on the notion of divisibility.

1. Definitions. We say that the Fibonacci number 8 is exactly divisible by 4 since $8 = 4 \times 2$. (Usually the word "exactly" is omitted.) Note that the following statements are equivalent:

(i) 8 is divisible by 4
(ii) 8 is a multiple of 4
(iii) 4 is a factor of 8
(iv) 4 is a divisor of 8
(v) 4 divides 8 (abbreviated: $4 \mid 8$).

Recalling that an integer is merely a whole number (0 included!), we may now say that a positive integer greater than one is a prime if and only if it has exactly two divisors. Thus F_3, F_4, and F_5 are primes, but $F_6 = 8$ is composite since it has four factors: 1, 2, 4, 8. Note that 2 is the smallest prime--and the only even one. In the interest of concise statements, 1 is normally not considered a prime (although it shares the property of "being divisible only by 1 and itself"). I.e., if we called 1 a prime, many theorems in elementary number theory would have to be reworded.

Digressing a moment we might observe that the following is an unsolved problem.

Is there a composite Fibonacci number with exactly three divisors?

If there were, then it would have to be the square of a prime--but 1 and 144 are the only known square Fibonacci numbers. In Tomorrow's Math, C.S. Ogilvy reports that one investigator is close to a solution of this problem.

2. Tests for Divisibility. In order to prove that, say, the Fibonacci number 987 is composite, it suffices to find a single divisor n such that $1 < n < 987$. Now certainly 987 is not divisible by 2 since it doesn't end in an even digit (0, 2, 4, 6, 8). But we see that 3 divides 987. Perhaps you were able to reach this conclusion mentally by noting that $9 + 8 + 7 = 24$ = a multiple of 3. This procedure, commonly called casting out 3's, is one of several simple tests for divisibility. For convenience we list some of the simpler tests for divisibility.

N is divisible by 2 if and only if it ends in an even digit.

N is divisible by 3 if and only if the sum of the digits of N is a multiple of 3.

N is divisible by 4 if and only if the number consisting of the last two digits of N is a multiple of 4.

N is divisible by 5 if and only if N ends in "0" or "5".

N is divisible by 6 if and only if N is a multiple of both 2 and 3.

N is divisible by 8 if and only if the number consisting of the last three digits of N is a multiple of 8.

N is divisible by 9 if and only if the sum of the digits of N is a multiple of 9.

Many other tests are easy to formulate: N is a multiple of 10 if and only if N ends in "0"; N is a multiple of 12 if and only if both 3 and 4 are factors of N. Some caution, however, is desirable. A number can be divisible by both 3 and 6, and yet not be divisible by 18. (Find an example.)

3. Prime or Composite? Suppose that you are confronted with the problem of determining whether or not $F_{13} = 233$ is a prime. How much work is involved? Assume that you may use any of the previously mentioned tests "free of charge". How many additional questions of the form "Is 233 divisible by so-and-so?", must you ask?

Of course there is no need to check for divisibility by 6 (or any other composite number) for if 233 is not itself a prime, it will have to have a prime factor less than 233. Fortunately we need not try each of the 50 primes less than 233. The simple but remarkable fact is that we can get by with no more than three "questions" (assuming that we test for divisibility by 2, 3, and 5 mentally): Is 233 divisible by 7? 11? 13?

In the case of 233, it turns out that the answer to each of these questions is "no". Let us see why this means that 233 must be a prime. If 233 were composite, it would have to have a prime factor $p > 13$. Thus $q = 233/p$ would be an integer, greater than 1 but less than 17; i.e., q, and hence 233, would have to be divisible by at least one of 2, 3, 5, 7, 11 or 13--a contradiction.

4. Problems. Solutions to these problems may be found on page 64.

1.1. What do you notice about every third Fibonacci number? Every fourth? Every fifth?

1.2. Try to guess a generalization of problem 1.1.

1.3. It is desirable to be able to define the greatest common divisor d of two integers (not both zero) without using the word "greatest". Do this, using only the following words and symbols: d, a, b, k, $|$, and, if, then.

1.4. Using 987 as an example, explain why the test for divisibility by 9 works.

1.5. What is the minimum number of questions that you need to ask in order to determine whether or not $F_{19} = 4181$ is a prime?

1.6. Let S be the following set of numbers:

4	7	10	13	16	19	22	...
7	12	17	22	27	32	37	...
10	17	24	31	38	45	52	...
13	22	31	40	49	58	67	...
16	27	38	49	60	71	82	...
.	.	.	.	.	.	.	
.	.	.	.	.	.	.	

Prove that (a) if N is in S, then 2N + 1 is composite and (b) if N is not in S, then 2N + 1 is a prime. Assume that the numbers in each row form an arithmetic progression and that the first column is the same as the first row.

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REQUEST

The Fibonacci Bibliographical Research Center desires that any reader finding a Fibonacci reference send a card giving the reference and a brief description of the contents. Please forward all such information to:

Fibonacci Bibliographical Research Center,
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San Jose State College,
San Jose, Calif.

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THE FIBONACCI SEQUENCE AS IT APPEARS IN NATURE

S. L. Basin
San Jose State College

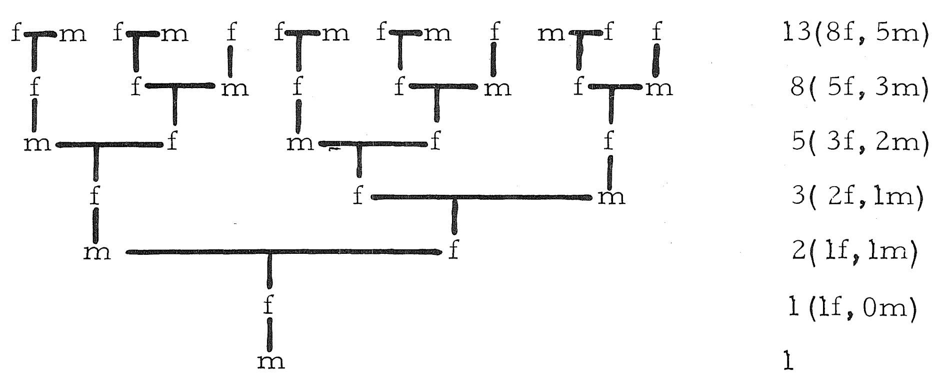
1. INTRODUCTION

The regular spiral arrangement of leaves around plant stalks has enjoyed much attention by botanists and mathematicians in their attempt to unravel the mysteries of this organic symmetry. Because of the abundance of literature on phyllotaxis no more attention will be devoted to it here. However, the Fibonacci numbers have the strange habit of appearing where least expected in other natural phenomena. The following snapshots will demonstrate this fact. (See references 1 and 3 for a discussion of phyllotaxis.)

2. THE GENEALOGICAL TREE OF THE MALE BEE

We shall trace the ancestral tree of the male bee backwards, keeping in mind that the male bee hatches from an unfertilized egg. The fertilized eggs hatch into females, either workers or queens.

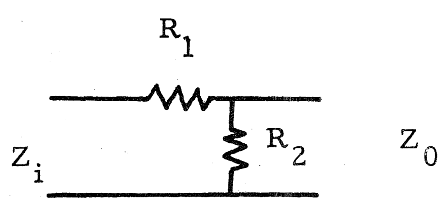
The following diagram clearly shows that the number of ancestors in any one generation is a Fibonacci number. The symbol (m) represents a male and the symbol (f) represents a female.



3. SIMPLE ELECTRICAL NETWORKS

Even those people interested in electrical networks cannot escape from our friend Fibonacci. Consider the following simple network of resistors known as a ladder network. This circuit consists of n L-sections in cascade and can be characterized or described by calculating the attenuation which is simply the input voltage divided by the output voltage and denoted by A, the input impedance Z_i and the output impedance Z_0. (For a detailed discussion refer to reference 4.)

Proceeding in a manner similar to mathematical induction, consider the following ladder networks.

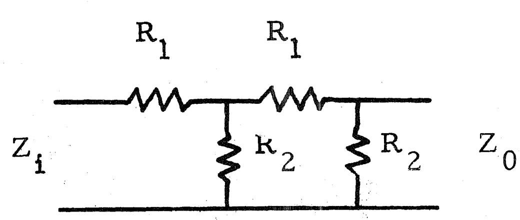


When $n = 1$, $Z_0 = R_2$

$Z_i = R_1 + R_2$

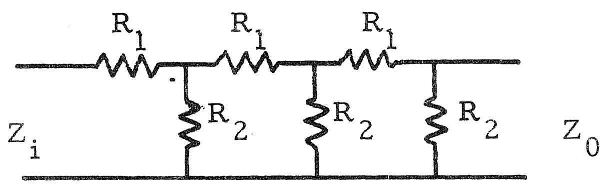
$A = R_1/R_2 + 1.$

When n = 2:



$$Z_0 = \frac{R_2(R_1 + R_2)}{R_1 + 2R_2}, \quad A = \frac{(R_1 + R_2)(R_1 + 2R_2) - R_2^2}{R_2^2}$$

$$Z_i = \frac{R_1R_2(R_1 + 2R_2) + R_2(R_1 + R_2)}{R_1 + 2R_2}$$



When n = 3:

$$Z_0 = \frac{R_1R_2(R_1 + 2R_2) + R_2^2(R_1 + R_2)}{(R_1 + R_2)(R_1 + 3R_2)}$$

$$Z_i = \frac{R_1^3 + 5R_1^2R_2 + 6R_1R_2^2 + R_2^3}{R_1^2 + 4R_1R_2 + 3R_2^2}$$

$$A = \frac{R_1^3 + 5R_1^2R_2 + 6R_1R_2^2 + R_2^3}{R_2^2}$$

Now suppose all the resistors have the same value, namely, $R_1 = R_2 = 1$ ohm. We have by induction:

$$Z_0 = \frac{F_{2n-1}}{F_{2n}}, \qquad A = (F_{2n-1} + F_{2n}) = F_{2n+1},$$

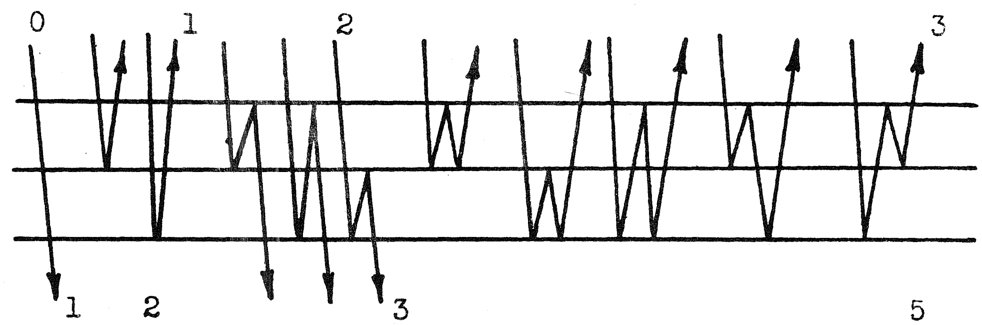
$$Z_i = \frac{F_{2n+1}}{F_{2n}}$$

In other words, the ladder network can be analyzed by inspection; as n is allowed to increase, n = 1, 2, 3, 4, . . . , the value of Z_0 for n L-sections coincides with the nth term in the sequence of Fibonacci ratios, i.e., 1/1, 2/3, 5/8, 13/21, ... The value for A is given by the sum of the numerator and denominator of Z_0. The value of Z_i is also clearly related to the expression for A and Z_0.

4. SOME REFLECTIONS (Communicated to us by Leo Moser)

The reflection of light rays within two plates of glass is expressed in terms of the Fibonacci numbers, i.e., if no reflections are allowed, one ray will emerge; if one reflection is allowed, two rays will emerge, ..., etc.

Number of reflections



Number of emerging rays

Refer to problem B-6 of Elementary Problems and Solutions.

For Additional Reading

1. Introduction to Geometry, H.S.M. Coxeter, pp. 169-172. John Wiley and Sons, Inc., 1961. A complete chapter on phyllotaxis and Fibonacci numbers appears in easily digestible treatment.

2. The Fibonacci Numbers, N.N. Vorobyov, Blaisdell, New York, 1961. (Translation from the Russian by Halina Moss)

This booklet discusses the elementary properties of Fibonacci numbers, their application to geometry, and their connection with the theory of continued fractions.

3. The Language of Mathematics, Robert Land, Chapter XIII, pp. 215-225, John Murray, 1960, London. A very interesting chapter including some phyllotaxis.

4. Appearance of Fibonacci Numbers and the Q Matrix in Electrical Network Theory, S.L. Basin, Mathematics Magazine, March, 1963.

EXPLORING FIBONACCI NUMBERS

Brother U. Alfred

What are currently known as Fibonacci numbers came into existence as part of a mathematical puzzle problem proposed by Leonardo Pisano (also known as Fibonacci) in his famous book on arithmetic, the Liber abaci (1202). He set up the following situation for the breeding of rabbits.

Suppose that there is one pair of rabbits in an enclosure in the month of January; that these rabbits will breed another pair of rabbits in the month of February; that pairs of rabbits always breed in the second month following birth and thereafter produce one pair of rabbits monthly. What is the number of pairs of rabbits at the end of December?

To solve this problem, let us set up a table with columns as follows:

(1) Number of pairs of breeding rabbits at the beginning of the given month;
(2) Number of pairs of non-breeding rabbits at the beginning of the month;
(3) Number of pairs of rabbits bred during the month;
(4) Number of pairs of rabbits at the end of the month.

MONTH	(1)	(2)	(3)	(4)
January	0	1	0	1
February	1	0	1	2
March	1	1	1	3
April	2	1	2	5
May	3	2	3	8
June	5	3	5	13
July	8	5	8	21
August	13	8	13	34
September	21	13	21	55
October	34	21	34	89
November	55	34	55	144
December	89	55	89	233

The answer to the original question is that there are 233 pairs of rabbits at the end of December. But the curious fact that characterizes the series of numbers evolved in this way is: any one number is the sum of the two previous numbers. Furthermore, it will be observed that all four columns in the above table are formed from numbers of the same series which has since come to be known as THE Fibonacci series: 0, 1, 1, 2, 3, 5, 8, 13, 21, ...

EXPLORATION

Did anybody ever find out what happened to the "Fibonacci rabbits" when they began to die? Since they have been operating with such mathematical regularity in other respects, let us assume the following as well. A pair of rabbits that is bred in February of one year breeds in April and every month thereafter including February of the following year. Then this pair of rabbits dies at the end of February.

(1) How many pairs of rabbits are there at the end of December of the second year?

(2) How many pairs of rabbits would there be at the end of n months, where n is greater than or equal to 12? (See what follows for notation.)

Assume that the original pair of rabbits dies at the end of December of the first year.

NAMES FOR ALL FIBONACCI NUMBERS

The inveterate Fibonacci addict tends to attribute a certain individuality to each Fibonacci number. Mention 13 and he thinks F_7; 55 and F_{10} flashes through his mind. But regardless of this psychological quirk, it is convenient to give the Fibonacci numbers identification tags and since they are infinitely numerous, these tags take the form of subscripts attached to the letter F. Thus 0 is denoted F_0;

the first 1 in the series is F_1; the second 1 is F_2; 2 is F_3; 3 is F_4; 5 is F_5; etc. The following table for F_n shows a few of the Fibonacci numbers and then provides additional landmarks so that it will be convenient for each Fibonacci explorer to make up his own table.

n	F_n	n	F_n
0	0	13	233
1	1	14	377
2	1	15	610
3	2	16	987
4	3	20	6765
5	5	30	832040
6	8	40	102334155
7	13	50	12586269025
8	21	60	1548008755920
9	34	70	190392490709135
10	55	80	23416728348467685
11	89	90	2880067194370816120
12	144	100	354224848179261915075

SUMMATION PROBLEMS

The first question we might ask is: What is the sum of the first n terms of the series? A simple procedure for answering this question is to make up a table in which we list the Fibonacci numbers in one column and their sum up to a given point in another.

n	F_n	Sum
1	1	1
2	1	2
3	2	4
4	3	7
5	5	12
6	8	20
7	13	33
8	21	54

What does the sum look like? It is not a Fibonacci number, but if we add 1 to the sum, it is the Fibonacci number two steps ahead. Thus we could write:

$$1 + 2 + 3 + \ldots\ldots + 34 + 55(= F_{10}) = 143 = 144(= F_{12}) - 1,$$

where we have indicated the names of some of the key Fibonacci numbers in parentheses. It is convenient at this point to introduce the summation notation. The above can be written more concisely:

$$\sum_{k=1}^{10} F_k = F_{12} - 1.$$

The Greek letter Σ (sigma) means: Take the sum of quantities F_k, where k runs from 1 to 10. We shall use this notation in what follows.

It appears that the sum of any number of consecutive Fibonacci numbers starting with F_1 is found by taking the Fibonacci number two steps beyond the last one in the sum and subtracting 1. Thus if we were to add the first hundred Fibonacci numbers together we would expect to obtain for an answer $F_{102} - 1$. Can we be sure of this? Not completely, unless we have provided some form of proof. We shall begin with a numerical proof meaning a proof that uses specific numbers. The line of reasoning employed can then be readily extended to the general case.

Let us go back then to the sum of the first ten Fibonacci numbers. We have seen that this sum is $F_{12} - 1$. Now suppose that we add 89 (or F_{11}) to both sides of the equation. Then on the lefthand side we have the sum of the first eleven Fibonacci numbers and on the right we have:

$$144(= F_{12}) - 1 + 89(= F_{11}) = 233(= F_{13}) - 1.$$

Thus, proceeding from the sum of the first ten Fibonacci numbers to the sum of the first eleven Fibonacci numbers, we have shown that the same type of relation must hold. Is it not evident that we could now go on from eleven to twelve; then from twelve to thirteen; etc., so that the relation must hold in general?

This is the type of reasoning that is used in the general proof by mathematical induction. We suppose that the sum of the first n Fibonacci numbers is $F_{n+2} - 1$. In symbols:

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1$$

We add F_{n+1} to both sides and obtain:

$$\sum_{k=1}^{n+1} F_k = F_{n+2} - 1 + F_{n+1} = F_{n+3} - 1$$

by reason of the fundamental property of Fibonacci series that the sum of any two consecutive Fibonacci numbers is the next Fibonacci number. We have now shown that if the summation holds for n, it holds also for n + 1. All that remains to be done is to go back to the beginning of the series and draw a complete conclusion. Let us suppose, as can readily be done, that the formula for the sum of the first n terms of the Fibonacci sequence holds for $n \leq 7$. Since the formula holds for seven, it holds for eight; since it holds for eight, it holds for nine; etc. etc. Thus the formula is true for all integral positive values of n.

We have seen from this example that there are two parts to our mathematical exploration. In the first we observe and conjecture and thus arrive at a formula. In the second we prove that the formula is true in general.

Let us take one more example. Suppose we wish to find the sum of all the odd-numbered Fibonacci numbers.

Again, we can form our table.

n	F_n	Sum
1	1	1
3	2	3
5	5	8
7	13	21
9	34	55
11	89	144
13	233	377

This is really too easy. We have come up with a Fibonacci number as the sum. Actually it is the very next after the last quantity added. We shall leave the proof to the explorer. However, the question of fitting the above results into notation might cause some trouble. What we need is a type of subscript that will give us just the odd numbers and no others. For the above sum to 13, we would write:

$$\sum_{k=1}^{7} F_{2k-1} = F_{14}$$

It will be seen that when k is 1, 2k-1 is 1; when k is 2, 2k-1 is 3; etc. and when k is 7, 2k-1 is 13. In general, the relation for the sum of the first n odd-subscript Fibonacci numbers would be:

$$\sum_{k=1}^{n} F_{2k-1} = F_{2n} .$$

EXPLORATION

1. Determine the sum of the first n even-subscript Fibonacci numbers.

2. If we take every fourth Fibonacci number and add, four series are possible:

   (a) Subscripts 1, 5, 9, 13, ...
   (b) Subscripts 2, 6, 10, 14, ...

(c) Subscripts 3, 7, 11, 15, . . .
(d) Subscripts 4, 8, 12, 16, . . .

Hint: Look for products or squares or near-products or near-squares of Fibonacci numbers as the result.

3. If we take every third Fibonacci number and add, three series are possible:

(a) Subscripts 1, 4, 7, 10, . . .
(b) Subscripts 2, 5, 8, 11, . . .
(c) Subscripts 3, 6, 9, 12, . . .

Hint: Double the sum and see whether you are near a Fibonacci number.

Until next issue, good hunting!

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RESEARCH PROJECT: FIBONACCI NIM

Consider a game involving two players in which initially there is a group of 100 or less objects. The first player may reduce the pile by any FIBONACCI NUMBER (member of the series 1, 1, 2, 3, 5, 8, 13, 21, . . .). The second player does likewise. The player who makes the last move wins the game.

(1) Is it always possible for the first player to win the game?
(2) If not, under what conditions can he be sure of winning?

****

1.1. Every third Fibonacci number is even; every fourth, a multiple of 3; every fifth, a multiple of 5.

1.2. Every nth Fibonacci number is divisible by F_n.

1.3. d is the greatest common divisor of a and b if and only if

(i) $d \mid a$, $d \mid b$, and

(ii) if $k \mid a$ and $k \mid b$, then $k \mid d$.

1.4.
$$\begin{aligned} 987 &= 9\times 100 + 8\times 10 + 7 \\ &= 9 \times (99 + 1) + 8 \times (9 + 1) + 7 \\ &= (9 \times 99) + \underline{9} + (8 \times 9) + \underline{8} + \underline{7} \\ &= 9 \times (99 + 8) + (9 + 8 + 7) \end{aligned}$$

Since the first term is a multiple of 9, 987 will be divisible by 9 if and only if $9 + 8 + 7$ is a multiple of 9.

1.5. One need only try the primes not exceeding the greatest integer equal to or less than the square root of 4181 or 64 +. How many primes are less than 64?

1.6. The number in the i-th row and j-th column is given by the formula

$$(2j + 1)\, i + j.$$

(a) If N is in **S**, then $2N + 1$ is of the form

$$2\{(2j+1)i + j\} + 1 = 4ij + 2i + 2j + 1 = (2i+1)(2j+1)$$
$$= \text{a composite number.}$$

(b) To prove that if N is not in **S**, then $2N + 1$ is a prime we consider an equivalent statement (called the contrapositive): If $2N + 1$ is not a prime, then N is in **S**.

Now if $2N + 1$ is not a prime, it has an odd factor $2i' + 1$ (which is between 1 and $2N + 1$). Thus

$$2N + 1 = (2j' + 1)(2i' + 1) = 2\{(2j' + 1)i' + j'\} + 1$$

or $N = (2j' + 1)i' + j'$, i.e., N lies in row i' and column j'.

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A PRIMER ON THE FIBONACCI SEQUENCE PART I

S.L. Basin and V.E. Hoggatt, Jr.

1. INTRODUCTION.

The proofs of Fibonacci identities serve as very suitable examples of certain techniques encountered in a first course in algebra. With this in mind, it is the intention of this series of articles to introduce the beginner to a few techniques in proving some number theoretic identities as well as furnishing examples of well-known methods of proof such as mathematical induction. The collection of proofs that will be given in this series may serve as a source of elementary examples for classroom use.

The use of matrix algebra in proving many theorems will be developed from basic principles in the next issue.

2. SOME SIMPLE PROPERTIES OF THE FIBONACCI SEQUENCE.

By observation of the sequence $\{1, 1, 2, 3, 5, 8, \ldots\}$, it is easily seen that each term is the sum of the two preceding terms. In mathematical language, we define this sequence by letting

$F_1 = 1$, $F_2 = 1$, and, for all integral n,

Definition (A) $\qquad F_{n+2} = F_{n+1} + F_n$ holds.

The first few Fibonacci numbers are:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ...

The Lucas Numbers, L_n, satisfy the same recurrence relation but have different starting values, namely,

$L_1 = 1$, $L_2 = 3$, and

Definition (B) $\qquad L_{n+2} = L_{n+1} + L_n$ holds.

The first few Lucas Numbers are:

1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ...

The following are some simple formulas which are called Fibonacci Number Identities or Lucas Number Identities for $n \geq 1$.

I. $F_1 + F_2 + F_3 + \ldots\ldots + F_n = F_{n+2} - 1$

II. $L_1 + L_2 + L_3 + \ldots\ldots + L_n = L_{n+2} - 3$

III. $F_{n+1} F_{n-1} - F_n^2 = (-1)^n$

IV. $L_{n+1} L_{n-1} - L_n^2 = 5(-1)^{n+1}$

V. $L_n = F_{n+1} + F_{n-1}$

VI. $F_{2n+1} = F_{n+1}^2 + F_n^2$

VII. $F_{2n} = F_{n+1}^2 - F_{n-1}^2$

VIII. $F_{2n} = F_n L_n$

IX. $F_{n+p+1} = F_{n+1} F_{p+1} + F_n F_p$

X. $F_1^2 + F_2^2 + \ldots\ldots F_n^2 = F_n F_{n+1}$

XI. $L_n^2 - 5 F_n^2 = 4(-1)^n$

XII. $F_{-n} = (-1)^{n+1} F_n$

XIII. $L_{-n} = (-1)^n L_n$

3. MATHEMATICAL INDUCTION

Any proofs of the foregoing identities ultimately depend upon the postulate of complete mathematical induction.

First one has a formula involving an integer n. For some values of n the formula has been seen to be true. This may be one, two, or say, twenty times. Now the excitement sets in....Is it true for all positive n? One may prove this by appealing to mathematical induction, whose three phases are:

A. Statement P(1) is true by trial. (If you can't find a first true case...why do you think it's true for any n let alone all n? Here you need some true cases to start with.)

An example of statement P(n) is

$$1 + 2 + 3 + \ldots\ldots + n = n(n+1)/2.$$

It is simple to see P(1) is true, that is

$$1 = 1(1+1)/2.$$

B. The truth of statement P(k) logically implies the truth of statement P(k + 1). In other words: If P(k) is true, then P(k + 1) is true. This step is commonly referred to as the inductive transition.

The actual method used to prove this implication may vary from simple algebra to very profound theory.

C. The statement that 'The proof is complete by mathematical induction.'

4. SOME ELEMENTARY FIBONACCI PROOFS.

Let us prove identity I.
Recall from (A) that $F_1 = 1$, $F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$

Statement P(n) is

$$F_1 + F_2 + F_3 + \ldots\ldots + F_n = F_{n+2} - 1$$

A. P(1) is true, since $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, so that
$F_1 = 1 = 2-1 = F_3 - 1.$

B. Assume P(k) is true, that is

$$F_1 + F_2 + \ldots\ldots\ldots\ldots + F_k = F_{k+2} - 1$$

From this we will show that the truth of P(k) demands the truth of P(k + 1), which is

$$(F_1 + F_2 + \ldots\ldots + F_k) + F_{k+1} = F_{k+3} - 1.$$

Since we assume P(k) is true, we may therefore assume that, in P(k + 1), we may replace $(F_1 + F_2 + \ldots\ldots + F_k)$ by $(F_{k+2} - 1)$. That is, P(k + 1) may be rewritten equivalently as $(F_{k+2} - 1) + F_{k+1} = (F_{k+2} + F_{k+1}) - 1 = F_{k+3} - 1.$ This is now clearly true from (A), which for n = k + 1 becomes $F_{k+3} = F_{k+2} + F_{k+1}$.

C. The proof is complete by mathematical induction.

5. A BIT OF THEORY (Cramer's Rule)

Given a second order determinant,

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

we have,

THEOREM I. $\begin{vmatrix} ax & b \\ cx & d \end{vmatrix} = x\,D, \quad -\infty < x < \infty .$

Proof: By definition

$$\begin{vmatrix} ax & b \\ cx & d \end{vmatrix} = (ax)\,d - b(cx) = x(ad-bc) = xD .$$

THEOREM II.

$$\begin{vmatrix} ax + by & b \\ cx + dy & d \end{vmatrix} = \begin{vmatrix} ax & b \\ cx & d \end{vmatrix} = xD, \quad \begin{matrix} -\infty < y < \infty, \\ -\infty < x < \infty . \end{matrix}$$

Proof: This is a simple exercise in algebra.

Suppose a system of two simultaneous equations possesses a unique solution (x_0, y_0), that is

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

is satisfied, if and only if, $x = x_0$, $y = y_0$.

This is specified by saying

$$\text{(C)} \qquad \begin{aligned} ax_0 + by_0 &= e \\ cx_0 + dy_0 &= f \end{aligned}$$

are true statements, with $D \neq 0$.

From our definition,

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

and from Theorem I, for $x = x_0$, we may write

$$x_0 D = \begin{vmatrix} ax_0 & b \\ cx_0 & d \end{vmatrix},$$

and from Theorem II, for $y = y_0$, $x = x_0$,

$$x_0 D = \begin{vmatrix} ax_0 + by_0 & b \\ cx_0 + dy_0 & d \end{vmatrix}.$$

But from (C) this may be rewritten

$$x_0 D = \begin{vmatrix} e & b \\ f & d \end{vmatrix}.$$

Thus,

$$x_0 = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{D} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$y_0 = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{D} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

which is Cramer's Rule.

We see this is true provided $D \neq 0$.
Determinant $D = 0$ if the two linear equations are inconsistent, (Graphs are distinct parallel lines.) or redundant (Graphs are the same line.). So that if the graphs (lines) are not parallel or coincident, then the common point of in-

tersection has the unique values of x_0 and y_0 as given by Cramer's Rule.

6. A CLEVER DEVICE IN ACHIEVING AN INDUCTIVE TRANSITION

From Definition (A) $F_1 = 1$, $F_2 = 1$ and

$$F_{n+2} = F_{n+1} + F_n$$

Suppose we write two examples of this (for $n = k$ and $n = k-1$).

$$F_{k+2} = F_{k+1} + F_k$$

$$F_{k+1} = F_k + F_{k-1}$$

Let us try to solve the pair of simultaneous linear equations.

$$\text{(D)} \qquad \begin{aligned} F_{k+2} &= x\,F_{k+1} + y\,F_k \\ F_{k+1} &= x\,F_k + y\,F_{k-1}\,. \end{aligned}$$

This is silly because we know the answer is $x_0 = 1$ and $y_0 = 1$, but using Cramer's Rule we note:

$$\text{(E)} \qquad y_0 = 1 = \frac{\begin{vmatrix} F_{k+1} & F_{k+2} \\ F_k & F_{k+1} \end{vmatrix}}{\begin{vmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{vmatrix}} = \frac{F_{k+1}^2 - F_k F_{k+2}}{F_{k+1}F_{k-1} - F_k^2}$$

Let us now use Mathematical Induction to prove identity III which is

$$P(n): \qquad F_{n+1}\,F_{n-1} - F_n^2 = (-1)^n$$

If we note that $F_0 = 0$ is valid, then

$$P(1): \quad F_2F_0 - F_1^2 = (-1)^1 = -1$$

is true. Thus part A is done.

Suppose P(k) is true. From (E),

$$1 = \frac{F_{k+1}^2 - F_kF_{k+2}}{F_{k+1}F_{k-1} - F_k^2}$$

so that

$$P(k+1): \quad F_{k+2}F_k - F_{k+1}^2 = (-1)^{k+1}$$

is indeed true!! Thus part B is done.

The proof is complete by mathematical induction and part C is done.

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SPECIAL NOTICE

The Fibonacci Association has on hand 22 copies of Dov Jarden, Recurrent Sequences, Riveon Lematematika, Jerusalem, Israel. This is a collection of papers on Fibonacci and Lucas numbers with extensive tables of factors extending to the 385th Fibonacci and Lucas numbers. The volume sells for $5.00 and is an excellent investment. Check or money order should be sent to Verner Hoggatt at San Jose State College, San Jose, Calif.

REQUEST

Maxey Brooke would like any references suitable for a Lucas bibliography. His address is 912 Old Ocean Ave., Sweeny, Tex.

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by S. L. Basin,
San Jose State College

Send all communications regarding Elementary Problems and Solutions to S. L. Basin, 946 Rose Ave., Redwood City, California. We welcome any problems believed to be new in the area of recurrent sequences as well as new approaches to existing problems. The proposer must submit his problem with solution in legible form, preferably typed in double spacing, with the name(s) and address of the proposer clearly indicated. Solutions should be submitted within two months of the appearance of the problems.

B-1. Proposed by I. D. Ruggles, San Jose State College, San Jose, Calif.

Show that the sum of twenty consecutive Fibonacci numbers is divisible by F_{10}.

B-2. Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Show that

$$u_{n+1} + u_{n+2} + \cdots\cdots + u_{n+10} = 11\, u_{n+7}$$

holds for generalized Fibonacci numbers such that $u_{n+2} = u_{n+1} + u_n$, where $u_1 = p$ and $u_2 = q$.

B-3. Proposed by J. E. Householder, Humboldt State College, Arcata, Calif.

Show that $F_{n+24} \equiv F_n \pmod 9$,

where F_n is the nth Fibonacci number.

B-4. Proposed by S. L. Basin and Vladimir Ivanoff, San Jose State College and San Carlos, Calif.

Show that $$\sum_{i=0}^{n} \binom{n}{i} F_i = F_{2n}.$$

Generalize.

B-5. Proposed by L. Moser, University of Alberta, Edmonton, Alberta.

Show that, with order taken into account, in getting paid an integral number n dollars, using only one-dollar and two-dollar bills, that the number of different ways is F_{n+1} where F_n is the nth Fibonacci number.

B-6. Proposed by L. Moser and M. Wyman, University of Alberta.

Light rays fall upon a stack of two parallel plates of glass, one ray goes through without reflection, two rays (one from each internal interface opposing the ray) will be reflected once but in different ways, three will be reflected twice but in different ways. Show that the number of distinct paths, which are reflected exactly n times, is F_{n+2}.

B-7. Proposed by H. W. Gould, West Virginia University, Morgantown, West Va.

Show that $$\frac{x(1-x)}{1-2x-2x^2+x^3} = \sum_{i=0}^{\infty} F_i^2 x^i .$$

Is the expansion valid at $x = 1/4$? That is, does

$$\sum_{i=0}^{\infty} F_i^2 / 4^i = 12/25 \ ?$$

B-8. Proposed by J. A. Maxwell, Stanford University.

Show that

$$\text{(i)} \quad F_{n+1} 2^n + F_n 2^{n+1} \equiv 1 \pmod 5$$

$$\text{(ii)} \quad F_{n+1} 3^n + F_n 3^{n+1} \equiv 1 \pmod{11}$$

$$\text{(iii)} \quad F_{n+1} 5^n + F_n 5^{n+1} \equiv 1 \pmod{29}.$$

Generalize.

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RESEARCH CONFERENCE of the FIBONACCI ASSOCIATION

On December 15, 1962, the Fibonacci Association held its first research conference at San Jose State College. Papers delivered on the occasion were as follows:

Some Determinants Involving Powers of Fibonacci Numbers . . . . Brother U. Alfred

Some Proofs of Conjectures in Brother Alfred's Paper . . . . .Terry Brennan

Squaring Rectangles Using Generalized Fibonacci Numbers . . . . Stanley L. Basin

The Period of the Ratio of Fibonacci Sequence Modulo M . . . . .John E. Vinson

Representations by Complete Fibonacci Sequences . . . . Verner E. Hoggatt, Jr.

Fibonacci Matrices . . . . . . James A. Maxwell

A Ray Incident on N Flats. . . . . . Bjarne Junge